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DIFFERENTIAL INDEPENDENCE OF MEROMORPHIC FUNCTIONS

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1. The differential field $\mathbb{M}$: Statement of problem and results

Historically, some complex functions, say polynomials $P(z)$, or sin z, e^z, etc., have been regarded as elementary (from the viewpoint of applications, human psychology, or pedagogy), while others, say the Euler Γ-function and the Riemann ζ-function have been assumed to be of a higher order of subtlety or complexity, see [2], [6]. Most attempts to formalize these ideas, within the ring of entire holomorphic functions - or, more likely, within its quotient field $\mathbb{M}$ of all meromorphic functions in the complex plane $\mathbb{C}$, see [9] [14] [16], have been based on the structural or axiomatic properties of $\mathbb{M}$ as a differential field. Namely, $\mathbb{M}$ has the structure of a commutative field, under the usual operations of addition $f+g$ and multiplication fg of meromorphic functions $f, g \in \mathbb{M}$; and further there is the derivation or differentiation operation $f \to f' = \frac{df}{dz}$ which is additive $(f+g)' = f' + g'$ and Leibnitzian $(fg)' = f'g + fg'$.

In this investigation of differential independence of meromorphic functions in $\mathbb{M}$, see Definition 1.1 below, we usually refer to the familiar description

$$\mathbb{M} = \{f \ : \ z \to f(z) \in \mathbb{C} \mid f \text{ holomorphic at all points } z \in \mathbb{C}, \text{ excepting the isolated poles of } f\} \tag{1.1}$$

and we shall introduce a hierarchy of complexity: starting from the subfield of constants

$$\mathbb{C} = \{f \in \mathbb{M} \mid f' = 0\}, \tag{1.2}$$

then the ring of polynomials and its quotient field of all rational functions

$$\mathbb{C}(z) = \{f \in \mathbb{M} \mid f \text{ has only a finite set of poles in } \mathbb{C}, \text{ and } \lim_{z\to\infty} f(z) \text{exists - either finite or } \infty \text{ (i.e. a pole)}\}, \tag{1.3}$$

and leading towards and then measuring the full complexity of $\mathbb{M}$ by means of the differential transcendence degree (see Definition 1.3 below),

$$\text{Diff-Trans } \partial^0\mathbb{M}/\mathbb{C} = \mathfrak{c} \ , \tag{1.4}$$

where $\mathfrak{c}$ is the cardinality of the continuum, [3]. For the proof of our principal result (1.4) in Theorem 2.3 below, our main task is the construction of a family $\{\Phi_r \mid r \in \mathbb{R}_+\}$

of entire functions, which are differential independent over $\mathbb{C}$, and this is carried out explicitly later in Section 2 of this paper.

We assume known the fundamental concepts and theory of abstract, i.e. axiomatic, differential fields, [5][7][8][10][11][16], and here we merely review briefly the relevant results and notations for the particular case of the differential field $\mathbb{M}$. For this immediate purpose we consider an arbitrary differential subfield $\mathbb{F}$ of $\mathbb{M}$, that is, $\mathbb{F}$ is a subfield of $\mathbb{M}$ and moreover $f \in \mathbb{F}$ implies $f' \in \mathbb{F}$, and we always assume that

$$\mathbb{C} \subset \mathbb{F} \subset \mathbb{M} ,$$

i.e. all differential subfields considered here contain the constant field $\mathbb{C}$. In this case a function $g \in \mathbb{M}$ is said to be *diff-algebraic* (differential algebraic) over $\mathbb{F}$ just in case: there exists a non-trivial polynomial expression (or polynomial form in the sense of algebra) P, with coefficients in $\mathbb{F}$ and in, say $\delta + 1 \geq 1$, unknowns or indeterminates $\{w^0, w^1, w^2, \ldots, w^\delta\}$ - so we can write $P(w^0, w^1, w^2, \ldots, w^\delta)$ - such that the meromorphic function

$$P(g(z), g'(z), g''(z), \ldots, g^{(\delta)}(z)) = 0 \quad \text{for all } z \in \mathbb{C}. \tag{1.5}$$

That is, g satisfies a polynomial (algebraic, possibly nonlinear) ordinary differential equation (1.5), with coefficients in $\mathbb{F}$ (not all zero) - and we note that the superscripts on $\{w^0, w^1, w^2, \ldots, w^\delta\}$ correspond to the orders of differentiation of g: NB they are not exponents.

Otherwise, when no such non-trivial polynomial expression exists, g is called *diff-transcendental* (differential transcendental) over $\mathbb{F}$.

Next consider two differential subfields of $\mathbb{M}$, say $\mathbb{F} \subset \mathbb{G}$ - so $\mathbb{F}$ is a subfield of $\mathbb{G}$, and $\mathbb{G}$ is an extension field of $\mathbb{F}$. If each function $g \in \mathbb{G}$ is diff-algebraic over $\mathbb{F}$, then $\mathbb{G}$ is called a *diff-algebraic extension* of $\mathbb{F}$ (or $\mathbb{G}$ is diff-algebraic over $\mathbb{F}$) - otherwise $\mathbb{G}$ is *diff-transcendental* over $\mathbb{F}$.

Here P (referred to in (1.5))is a polynomial expression (merely an appropriate array of coefficients, say in $\mathbb{F}$ or $\mathbb{C}$, as the case may be) as distinct from a polynomial function (say from $\mathbb{C}^{\delta+1}$ into $\mathbb{C}$). However, we often refer to P as a polynomial when the meaning is clear by the notation or context - in order to avoid awkward phraseology. In some references [8] P is designated as a "*differential polynomial*" form $P(w^0)$, in terms of the "*differential indeterminate*" w^0.

Remark 1.1. Let $\mathbb{F}$ be a differential subfield of $\mathbb{M}$, and we consider the adjunction of an element $g \in \mathbb{M}$ to construct the differential field $\mathbb{F} < g >$, namely the smallest differential field in $\mathbb{M}$ which contains $\mathbb{F}$ and g. That is, $\mathbb{F} < g >$ is the intersection of all differential subfields of $\mathbb{M}$ which each contains $\mathbb{F}$ and g (and hence g', g'', etc.). From a constructive viewpoint, an element $h \in \mathbb{M}$ belongs to $\mathbb{F} < g >$ just in case h is equal to some rational expression, with coefficients in $\mathbb{F}$, involving g and some finite number of the derivatives of g - since this set of elements clearly constitutes a differential subfield of $\mathbb{M}$ containing $\mathbb{F}$ and g.

From such a constructive description of $\mathbb{F} < g >$ we recognize that this field is the same as the countable adjunction $\mathbb{F}(g, g', g'', \dots)$, considering these fields without reference to differentiation. Then it is straightforward to demonstrate that $\mathbb{F} < g >$ is diff-algebraic over $\mathbb{F}$ if and only if $\mathbb{F}(g, g', g'', \dots)$ has a finite transcendence degree over $\mathbb{F}$ (in the usual sense of the algebra of field extensions, see [19], [20]).

As a consequence of this remark we note that $\mathbb{F} < g >$ is diff-algebraic over $\mathbb{F}$ if and only if g is diff-algebraic over $\mathbb{F}$, see [7], [8], [11]. As a trivial example we observe that $\mathbb{C} < z >= \mathbb{C}(z)$ is diff-algebraic over $\mathbb{C}$.

As before, a finite adjunction

$$\mathbb{G} = \mathbb{F} < g_1, g_2, \dots, g_N > := \mathbb{F} < g_1, g_2, \dots, g_{N-1} >< g_N > \tag{1.6}$$

is diff-algebraic over $\mathbb{F}$ if and only if $\mathbb{G}$ has a finite transcendence degree over the field $\mathbb{F}$; and this happens if and only if every g_j is diff-algebraic over $\mathbb{F}$, for $j = 1, \dots, N$. More generally we define the differential field $\mathbb{F} < g_\alpha >$, for the prescribed family $\{g_\alpha \in \mathbb{M} \mid \alpha \in A, \text{ an index set}\}$, as the smallest differential subfield of $\mathbb{M}$ containing $\mathbb{F}$ and every element g_α, for $\alpha \in A$. Then $\mathbb{F} < g_\alpha >$ (abbreviating for $F < g_\alpha | \alpha \in A >$) is diff-algebraic over $\mathbb{F}$ if and only if every g_α is diff-algebraic over $\mathbb{F}$, see [7] and [8, Ch II, Section 8].

Using these ideas, by familiar methods, we arrive at the general result for differential subfields of $\mathbb{M}$,

$$\mathbb{F} \subset \mathbb{G}_1 \subset \mathbb{G}_2; \tag{1.7}$$

namely, (see [8, Ch II, Section 8] and [11]):

If $\mathbb{G}_2$ is diff-algebraic over $\mathbb{G}_1$ and $\mathbb{G}_1$ is diff-algebraic over $\mathbb{F}$,
then $\mathbb{G}_2$ is diff-algebraic over $\mathbb{F}$.

The following definition makes precise the concept of the *differential independence* of a function family $\{g_\alpha \in \mathbb{M} \mid \alpha \in A, \text{an index set}\}$, over a differential subfield $\mathbb{F} \subset \mathbb{M}$.

Definition 1.1. *A (non-empty) finite set of functions of $\mathbb{M}$, say $\{g_1, g_2, \dots, g_N\}$ for some $N \geq 1$, is diff-independent (differential independent), over a given differential subfield $\mathbb{F} \subset \mathbb{M}$, in case:*

there is no (non-trivial) polynomical expression P, with coefficients in $\mathbb{F}$ and in $\sum_{j=1}^{N}(1 + \delta_j)$ indeterminates $w_j^m, j = 1, \dots, N$ and $m = 0, 1, \dots, \delta_j$, $(\delta_j \geq 0)$, say,

$$P(w_1^0, w_1^1, w_1^2, \dots, w_1^{\delta_1}, w_2^0, w_2^1, \dots, w_2^{\delta_2}, \dots, w_N^0, w_N^1, \dots, w_N^{\delta_N}), \tag{1.8}$$

such that the meromorphic function

$$P(g_1(z), g_1'(z), g_1''(z), \dots, g_1^{(\delta_1)}(z), g_2(z), g_2'(z), \dots, g_2^{(\delta_2)}(z), \dots,$$
$$g_N(z), g_N'(z), \dots, g_N^{(\delta_N)}(z)) \equiv 0 \tag{1.9}$$

for all $z \in \mathbb{C}$.

More generally, a family $\{g_\alpha | \alpha \in A,$ *an index set*$\}$ *of functions* $g_\alpha \in \mathbb{M}$ *for* $\alpha \in A$, *is diff-independent over* $\mathbb{F}$ *just in case:*

> *each (non-empty) finite subset of functions of this family is diff-independent over* $\mathbb{F}$.

Remark 1.2. In other words, for a diff-independent family $\{g_\alpha | \alpha \in A\}$ the only polynomial expression P of the form (1.8) that is annulled by a finite subfamily of these functions is the trivial polynomial with all coefficients zero. On the other hand, if there exists a non-trivial polynomial P which is annulled by some finite subfamily, then $\{g_\alpha | \alpha \in A\}$ is called *diff-dependent* over $\mathbb{F}$.

We specifically emphasize that for the indeterminates w_j^m in (1.8) above, the upper index relates to the order of differentiation of $g_j^{(m)}$, and is not an exponent. This observation is emphasized by the occasional reference, see [8], to P in (1.8) as a differential polynomial form involving the differential indeterminates $\{w_1^0, w_2^0, \dots, w_N^0\}$.

Remark 1.3. Let $\{g_\alpha | \alpha \in A\}$ be a family of functions, $g_\alpha \in \mathbb{M}$ for $\alpha \in A$, which is diff-independent over a differential subfield $\mathbb{F} \in \mathbb{M}$. Then the functions of the family are different, that is,

$$g_{\alpha_1} \neq g_{\alpha_2} \text{ for } \alpha_1 \neq \alpha_2 \in A.$$

Moreover, each of these functions is itself diff-transcendental over $\mathbb{F}$.

Remark 1.4. A family of functions $\{g_\alpha | \alpha \in A\}$ in $\mathbb{M}$ is diff-independent over the differential field $\mathbb{C}(z)$ if and only if:

For each finite subfamily, say $\{g_1, g_2, \dots, g_N\}$, there exists no non-trivial polynomial expression P, with coefficients in $\mathbb{C}$ and in $1 + \sum_{j=1}^{N}(1 + \delta_j)$ indeterminates, z and w_j^m (for $j = 1, 2 \dots, N$ and $m = 0, 1, \dots \delta_j$, with $\delta_j \geq 0$), say

$$P(z, w_1^0, w_1^1, w_1^2, \dots, w_1^{\delta_1}, w_2^0, w_2^1, \dots, w_2^{\delta_2}, \dots, w_N^0, w_N^1, \dots, w_N^{\delta_N}), \tag{1.10}$$

such that the meromorphic function

$$P(z, g_1(z), g_1'(z), g_1''(z), \dots, g_1^{(\delta_1)}(z), g_2(z), g_2'(z), \dots, g_2^{(\delta_2)}(z), \dots,$$
$$g_N(z), g_N'(z), \dots, g_N^{(\delta_N)}(z)) = 0 \tag{1.11}$$

for all $z \in \mathbb{C}$.

This Remark 1.4 follows immediately from Definition 1.1, with the field $\mathbb{F} = \mathbb{C}(z) = \mathbb{C} < z >$, provided we observe that a non-trivial polynomial expression P, as in (1.10), can also be regarded as a non-trivial polynomial expression, as in (1.8) above with coefficients that are polynomials of $\mathbb{C}(z)$, involving the remaining indeterminates w_j^m for $j = 1, 2, \ldots, N$ and $m = 0, 1, \ldots, \delta_j$.

Now consider the case where the differential subfield $\mathbb{G} \subset \mathbb{M}$ is a diff-transcendental extension of a prescribed differential field $\mathbb{F}$. Then it is known, see [8, Ch II, Section 9] and [11], that there exists a *differential transcendence basis* for $\mathbb{G}$ over $\mathbb{F}$, as in the next definition.

Definition 1.2. *Let $\mathbb{G}$ be a differential subfield of $\mathbb{M}$ and a diff-transcendental extension of the differential field $\mathbb{F}$, so*

$$\mathbb{C} \subset \mathbb{F} \subset \mathbb{G} \subset \mathbb{M}.$$

Then a family of functions $\{g_\alpha \in \mathbb{G} \ |\alpha \in A\}$ is a diff-transcendence basis for $\mathbb{G}$ over $\mathbb{F}$ in case:

(a) *the family $\{g_\alpha | \alpha \in A\}$ is diff-independent over $\mathbb{F}$,*
and

(b) *the family $\{g_\alpha | \alpha \in A\}$ diff-spans $\mathbb{G}$ in the sense that $\mathbb{G}$ is diff-algebraic over $\mathbb{F} < g_\alpha >$ (abbreviation for $\mathbb{F} < g_\alpha | \alpha \in A >$).*

In connection with the concept of a diff-transcendence basis of $\mathbb{G}$ over $\mathbb{F}$, it is further known that, [8, Ch II Section 9]:

(i) *each family $\{ g_\beta \in \mathbb{G} | \beta \in B\}$, which is diff-independent over $\mathbb{F}$, can be enlarged (augmented) to constitute a diff-transcendence basis for $\mathbb{G}$ over $\mathbb{F}$;*

(ii) *any such family of functions of $\mathbb{G}$, which diff-spans $\mathbb{G}$ over $\mathbb{F}$ as in (b) above, must contain a diff-transcendence basis for $\mathbb{G}$ over $\mathbb{F}$.*

From these two observations (i) and (ii) it is straightforward to prove, within ZFC-set theory, that two diff-transcendence bases for $\mathbb{G}$ over $\mathbb{F}$, as in Definition 1.2 above, must have the same cardinality.

Definition 1.3. *Let $\mathbb{G}$ be a differential subfield of $\mathbb{M}$ and a diff-transcendental extension of the differential field $\mathbb{F}$, so*

$$\mathbb{C} \subset \mathbb{F} \subset \mathbb{G} \subset \mathbb{M}. \tag{1.12}$$

Let $\{g_\alpha \in \mathbb{G} \ |\alpha \in A\}$ be a diff-transcendence basis for $\mathbb{G}$ over $\mathbb{F}$ as in Definition 1.2 above.

Then the diff-transcendence degree of $\mathbb{G}$ over $\mathbb{F}$ is the cardinal number

$$\textit{Diff-Trans } \partial^0 \mathbb{G}/\mathbb{F} = \textit{card } A. \tag{1.13}$$

As observed above, this diff-transcendence degree does not depend on the choice of the diff-transcendence basis for $\mathbb{G}$ over $\mathbb{F}$.

Further, if $\mathbb{G}$ *is diff-algebraic over* $\mathbb{F}$ *we define*

$$\text{(1.14)} \qquad \textit{Diff-Trans } \partial^0\mathbb{G}/\mathbb{F} = 0.$$

Remark 1.5. Consider differential subfields of $\mathbb{M}$

$$\text{(1.15)} \qquad \mathbb{C} \subset \mathbb{F} \subset \mathbb{G}_1 \subset \mathbb{G}_2 \subset \mathbb{M}.$$

Then it follows from Definitions 1.2 and 1.3, in particular the items (i) and (ii) above, that (see [8, Ch II Section 9] and [11]):

$$\text{(1.16)} \qquad \text{Diff-Trans } \partial^0\mathbb{G}_2/\mathbb{F} = \text{Diff-Trans } \partial^0\mathbb{G}_2/\mathbb{G}_1 + \text{ Diff-Trans } \partial^0\mathbb{G}_1/\mathbb{F}.$$

With these fundamental concepts and preliminaries in place, we now present an intrinsic definition of the elementary functions $\mathbb{E}$ of $\mathbb{M}$, and formulate our principal result concerning the diff-transcendence degree of the differential field $\mathbb{M}$ over the constant field $\mathbb{C}$, and also over the diff-elementary field $\mathbb{E}$, compare [5], [15].

Definition 1.4. *In the differential field* $\mathbb{M}$ *define the set* $\mathbb{E}$ *of diff-elementary functions by*

$$\text{(1.17)} \qquad \mathbb{E} := \{f \in \mathbb{M} \mid f \textit{ diff-algebraic over } \mathbb{C}\}.$$

Theorem 1.1. *The set* $\mathbb{E}$ *of diff-elementary functions in* $\mathbb{M}$ *is a differential subfield, which is diff-algebraic over* $\mathbb{C}$*, and hence over* $\mathbb{C}(z)$*. Further, if* $h \in \mathbb{M}$ *is diff-algebraic over* $\mathbb{E}$*, then* $h \in \mathbb{E}$*.*

Proof. Certainly $\mathbb{C}$ and $\mathbb{C}(z)$ both lie in $\mathbb{E}$, see Remark 1.1 above. Now take $f, g \in \mathbb{E}$ and consider the differential field, $\mathbb{C} < f, g >$, which is diff-algebraic over $\mathbb{C}$. Then $f' \in \mathbb{C} < f, g >$, which is diff-algebraic over $\mathbb{C}$ and hence $f' \in \mathbb{E}$. Similarly, the rational combinations of f and g lie in $\mathbb{C} < f, g >$ and hence lie in $\mathbb{E}$. Therefore $\mathbb{E}$ is a differential field, and further $\mathbb{E}$ is diff-algebraic over $\mathbb{C}$.

Let $h \in \mathbb{M}$ be diff-algebraic over $\mathbb{E}$. Then $\mathbb{E} < h >$ is diff-algebraic over $\mathbb{E}$ and hence over $\mathbb{C}$. Thus h is diff-algebraic over $\mathbb{C}$, so $h \in \mathbb{E}$. □

The next lemma asserts that if $g \in \mathbb{M}$ is diff-transcendental over any one of the differential fields $\mathbb{E}$, $\mathbb{C}(z), \mathbb{C}$, then it is also diff-transcendental over each of these three differential fields. (An alternative version of this assertion merely replaces the condition "diff-transcendental" by "diff-algebraic" throughout). In fact, we here present a much stronger result.

Lemma 1.1. *If a family* $\{g_\alpha | \alpha \in A\}$ *of functions* $g_\alpha \in \mathbb{M}$*, for* $\alpha \in A$ *an index set, is diff-independent over any one of the differential fields* $\mathbb{E}$*,* $\mathbb{C}(z)$*, or* $\mathbb{C}$*, then this family is diff-independent over every one of these three differential fields.*

Proof. Assume the family $\{g_\alpha | \alpha \in A\}$ is diff-independent over $\mathbb{E}$. Then, a fortiori, the family is diff-independent over $\mathbb{C}(z)$ and also over $\mathbb{C}$. In the same way, if $\{g_\alpha | \alpha \in A\}$ is diff-independent over $\mathbb{C}(z)$, then it is diff-independent over $\mathbb{C}$.

Therefore, the lemma will be proved if we show that a family $\{g_\alpha | \alpha \in A\}$ that is diff-independent over $\mathbb{C}$, is necessarily diff-independent over $\mathbb{E}$. We proceed by a contradiction argument and suppose that $\{g_\alpha | \alpha \in A\}$ is not diff-independent over $\mathbb{E}$. In such a case there exists some finite subfamily, say $\{g_1, g_2, \ldots, g_N\}$ which is not diff-independent over $\mathbb{E}$, and so there exists a non-trivial polynomial P, with coefficients in $\mathbb{E}$, which is nullified upon replacing the unknowns by the functions $g_1, g_2, \ldots, g_N$, and their derivatives, as in Definition 1.1 (1.8) and (1.9).

Say g_N, or its derivatives, appear in P. Then, g_N is diff-algebraic over the differential field $\mathbb{E} < g_1, g_2, \ldots, g_{N-1} >$, and so

$$\text{(1.18)} \qquad \text{Diff-Trans } \partial^0 E < g_1, g_2, \ldots, g_N > /\mathbb{E} \leq N - 1.$$

From (1.16) of Remark 1.5 above, we then observe that

$$\text{(1.19)} \qquad \begin{aligned} &\text{Diff-Trans } \partial^0 \mathbb{E} < g_1, g_2, \ldots, g_N > /\mathbb{C} = \\ &\text{Diff-Trans } \partial^0 \mathbb{E} < g_1, g_2, \ldots, g_N > /\mathbb{E} + \text{ Diff-Trans } \partial^0 \mathbb{E}/\mathbb{C}. \end{aligned}$$

However $\mathbb{E}$ is diff-algebraic over $\mathbb{C}$, and our supposition then leads to

$$\text{(1.20)} \qquad \begin{aligned} N \leq &\text{ Diff - Trans } \partial^0 \mathbb{E} < g_1, \ldots, g_N/\mathbb{C} = \\ &\text{Diff-Trans } \partial^0 \mathbb{E} < g_0, \ldots, g_N > /\mathbb{E} \leq N - 1 \ , \end{aligned}$$

which is impossible.

Therefore we conclude that $\{g_1, g_2, \ldots, g_N\}$ being diff-independent over $\mathbb{C}$ implies that this same subfamily of N functions is diff-independent over $\mathbb{E}$, and hence the lemma is proved. □

Remark 1.6. In the situation of Lemma 1.1 we often refer to the family $\{g_\alpha | \alpha \in A\}$ as diff-independent (over $\mathbb{C}$ being understood).

Theorem 1.2. *Let $\mathbb{G}$ be a differential subfield of $\mathbb{M}$, and assume that $\mathbb{E} \subset \mathbb{G}$. Then*

$$\text{(1.21)} \qquad \textit{Diff-Trans } \partial^0 \mathbb{G}/\mathbb{E} = \textit{ Diff-Trans } \partial^0 \mathbb{G}/\mathbb{C}(z) = \textit{ Diff-Trans } \partial^0 \mathbb{G}/\mathbb{C}.$$

Proof. If $\mathbb{G}$ is diff-algebraic over any one of $\mathbb{E}$, $\mathbb{C}(z)$, or $\mathbb{C}$, then it is also diff-algebraic over every one of these three differential fields, and, in this case, each term in (1.21) is zero, see Lemma 1.1. Thus we need only consider the case where $\mathbb{G}$ is diff-transcendental over each of the three differential fields $\mathbb{E}$, $\mathbb{C}(z)$, and $\mathbb{C}$.

Let $\{g_\alpha | \alpha \in A\}$, with $g_\alpha \in \mathbb{G}$ for each $\alpha \in A$, an index set as before, be a diff-transcendence basis for $\mathbb{G}$ over $\mathbb{C}$. We shall demonstrate that $\{g_\alpha | \alpha \in A\}$ is also a transcendence basis for $\mathbb{G}$ over $\mathbb{C}(z)$, and equally well for $\mathbb{G}$ over $\mathbb{E}$. This will then verify that each term in (1.21) is the same cardinal number, namely card A.

By Lemma 1.1 the family $\{g_\alpha | \alpha \in A\}$ is diff-independent over $\mathbb{C}(z)$, and also over $\mathbb{E}$. Since $\{g_\alpha | \alpha \in A\}$ is a diff-transcendence basis for $\mathbb{G}$ over $\mathbb{C}$, $\mathbb{G}$ is diff-algebraic over

$\mathbb{C} < g_\alpha >$, (abbreviation for $\mathbb{C} < g_\alpha | \alpha \in A >$). But then, a fortiori, $\mathbb{G}$ is diff-algebraic over $\mathbb{C} < z, g_\alpha >$, and also over $\mathbb{E} < g_\alpha >$, because

$$\mathbb{C} < g_\alpha > \subset \mathbb{C} < z, g_\alpha > \subset \mathbb{E} < g_\alpha > \ . \tag{1.22}$$

Thus $\{g_\alpha | \alpha \in A\}$ is a diff-transcendence basis for $\mathbb{G}$ over $\mathbb{C}(z) = \mathbb{C} < z >$, and also over $\mathbb{E}$. Therefore the equalities of (1.21) are verified, and the theorem is proved. □

By Theorem 1.2 we know that

$$\text{Diff-Trans } \partial^0\mathbb{M}/\mathbb{E} = \text{ Diff-Trans } \partial^0\mathbb{M}/\mathbb{C}(z) = \text{Diff-Trans } \partial^0\mathbb{M}/\mathbb{C}. \tag{1.23}$$

Recall that card $\mathbb{M} = \mathfrak{c}$, the cardinality of the continuum, see [3]. This is easily established because each entire holomorphic function is determined by a complex power series, so the cardinality of the ring of entire functions is $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$, and the same cardinality describes the quotient field $\mathbb{M}$. Therefore we conclude that

$$\text{Diff-Trans } \partial^0\mathbb{M}/\mathbb{C} \leq \mathfrak{c} \ . \tag{1.24}$$

The earlier formula (1.4) asserts that equality holds in (1.24) above. This assertion will be proved in our principal Theorem 2.3 in Section 2 below.

We close this introductory Section 1 with some examples of diff-elementary functions, in the sense of Definition 1.4, and some diff-transcendental functions and moreover diff-independent sets of such functions in the differential field $\mathbb{M}$.

Example 1.1. We present here an illuminating list of diff-elementary functions, with the corresponding differential polynomials which they nullify in the sense of (1.5) or (1.11):

(*i*)	Polynomial $p \in \mathbb{C}(z)$ of degree $\delta \geq 0$	;	$w^{\delta+1}$ (i.e. $p^{(\delta+1)}(z) \equiv 0$)
(*ii*)	$\sin az$ for $a \neq 0$ in $\mathbb{C}$	;	$w^2 + a^2 w^0$
(*iii*)	e^{az} for $a \neq 0$ in $\mathbb{C}$	;	$w^1 - a w^0$
(*iv*)	$J_k(z)$ Bessel function of order $k \geq 0$	;	$z^2 w^2 + z w^1 + (z^2 - k^2) w^0$
(*v*)	$\mathfrak{p}(z)$, Weierstiass $\mathfrak{p}$-function,	;	$(w^1)^2 - 4(w^0)^3 + g_2 w^0 - g_3$ with invariants $g_2, g_3 \in \mathbb{C}$.

In the Appendix at the end of this paper we discuss the celebrated theorem of Hölder [2][4] proving that the Euler Γ-function [1][6] is not in $\mathbb{E}$, but is diff-transcendental over $\mathbb{C}$ - see [10] for a similar result for the Riemann ζ-function [6][13]. We further extend these results to demonstrate that the pairs

$$\Gamma(z), \Gamma(\sin 2\pi z) \qquad \text{and also} \qquad \Gamma(z), \zeta(\sin 2\pi z)$$

are each diff-independent over $\mathbb{C}$, that is, say,

$$\text{Diff-Trans } \ \partial^0\mathbb{C} < \Gamma(z), \zeta(\sin 2\pi z) > /\mathbb{C} = 2. \tag{1.25}$$

It is an unsolved problem as to whether the pair $\Gamma(z), \zeta(z)$ is diff-independent over $\mathbb{C}$.

2. Diff-independent sets of entire functions

In order to construct a diff-independent set of functions $\{\Phi_r \mid r > 0\}$, as mentioned earlier in Section 1, we shall provide a general existence theorem for entire holomorphic functions with prescribed values for finitely many derivatives at a given sequence of distinct points $a_\nu \in \mathbb{C}$, for $\nu \in \mathbb{N} = \{1, 2, 3, \dots\}$; an infinite set without accumulation points in $\mathbb{C}$.

Theorem 2.1. *Consider an infinite point set in the complex plane $\mathbb{C}$, with no accumulation points; so we can then enumerate these points by an injection $\mathbb{N} \to \mathbb{C}$, $\nu \to a_\nu$ and the resulting sequence $\{a_\nu\}$ satisfies $\lim_{\nu\to\infty} |a_\nu| = \infty$.*

For each $\nu \in \mathbb{N}$ let there be assigned an integer $\delta_\nu \geq 0$ and a corresponding complex function Δ_ν with domain $\operatorname{Dom}\Delta_\nu \subset [0, 1, 2, \dots, \delta_\nu]$, so

$$\Delta_\nu \ : \ \text{Dom } \Delta_\nu \to \mathbb{C}\ , \tag{2.1}$$

given by $\nu \to \Delta_\nu(j) = c_{\nu,j}$ for $j \in \operatorname{Dom}\Delta_j$. Then there exists an entire holomorphic function Φ, or

$$z \to \Phi(z) \quad \textit{for} \quad z = \mathbb{C}, \tag{2.2}$$

such that: for each $\nu \in \mathbb{N}$

$$\Phi(a_\nu) = c_{\nu,0}, \quad \Phi'(a_\nu) = c_{\nu,1}, \dots, \Phi^{(\delta_\nu)}(a_\nu) = c_{\nu,\delta_\nu}, \tag{2.3}$$

where

$$c_{\nu,j} = \Delta_\nu(j) \quad \textit{for} \quad j = 0, 1, \dots, \delta_\nu \tag{2.4}$$

(and we assign $c_{\nu,j} = 0$ for $j \notin$ Dom Δ_ν).

Proof. The celebrated factor-theorem of Weierstrass, see [6], asserts that there exists an entire holomorphic function H in $\mathbb{C}$, with zeros of order $\delta_\nu + 1$ at each point a_ν for $\nu \in \mathbb{N}$, and furthermore H has no other zeros in $\mathbb{C}$. Fix one such entire function H, and in a neighborhood of $z = a_\nu$ there is the convergent expansion

(2.5)
$$\begin{aligned} H(z) \quad &= \quad h_{\nu,1}[(z-a_\nu)^{\delta_\nu+1} + h_{\nu,2}(z-a_\nu)^{\delta_\nu+2} + \dots + h_{\nu,\delta_\nu+1}(z-a_\nu)^{2\delta_\nu+1}] + \\ &\qquad +\mathcal{O}(|z-a_\nu|^{2\delta_\nu+2}), \end{aligned}$$

where the constant $h_{\nu,1} \neq 0$, for each $\nu \in \mathbb{N}$.

Further, the partial-fraction-theorem of Mittag-Leffler, see [6], asserts that there exists a meromorphic function M in $\mathbb{C}$, with poles at a_ν for $\nu \in \mathbb{N}$, where the principal part is

$$\frac{1}{h_{\nu,1}}\left[\frac{\gamma_{\nu,0}}{(z-a_\nu)^{\delta_\nu+1}} + \frac{\gamma_{\nu,1}}{(z-a_\nu)^{\delta_\nu}} + \dots + \frac{\gamma_{\nu,\delta_\nu}}{(z-a_\nu)}\right], \tag{2.6}$$

and M has no other poles in $\mathbb{C}$. Here the complex numbers $\gamma_{\nu,0}, \gamma_{\nu,1}, \dots, \gamma_{\nu,\delta_\nu}$ are defined by the system of linear equations (where the subscript ν is suppressed for clarity of presentation).

$$\begin{aligned} \gamma_0 &= c_0 \\ \gamma_1 + h_2\gamma_0 &= c_1 \\ \gamma_2 + h_2\gamma_1 + h_3\gamma_0 &= c_2/2! \\ &\vdots \\ \gamma_\delta + h_2\gamma_{\delta-1} + \dots + h_{\delta+1}\gamma_0 &= c_\delta/\delta! \quad . \end{aligned} \tag{2.7}$$

Thus, for z in a neighborhood of a_ν, we have

$$M = \frac{1}{h_{\nu,1}}\left[\frac{\gamma_{\nu,0}}{(z-a_\nu)^{\delta_\nu+1}} + \dots + \frac{\gamma_{\nu,\delta_\nu}}{(z-a_\nu)}\right] + \text{ analytic function.} \tag{2.8}$$

Fix the choice of M and then define the meromorphic function, for $z \in \mathbb{C}$,

$$\Phi(z) = H(z)M(z), \tag{2.9}$$

which has no poles, excepting poles that might exist at the points $a_\nu \in \mathbb{C}$, for $\nu \in \mathbb{N}$. In this respect examine the function Φ in a neighborhood of a_ν. Here we have the convergent expansion (again suppressing the subscript ν for clarity)

(2.10)

$$\Phi(z) = \gamma_0 + (\gamma_1 + h_2\gamma_0)(z-a) + \dots + (\gamma_\delta + h_2\gamma_{\delta-1} + \dots + h_\delta\gamma_1 + h_{\delta+1}\gamma_0)(z-a)^\delta + \dots .$$

That is, near a_ν we have

$$\Phi(z) = c_{\nu,0} + c_{\nu,1}(z-a_0) + \frac{c_{\nu,2}}{2!}(z-a_\nu)^2 + \dots + \frac{c_{\nu,\delta_\nu}}{(\delta)!}(z-a_\nu)^{\delta_\nu} + \mathcal{O}(|z-a_j|^{\delta_\nu+1}), \tag{2.11}$$

so $\Phi(z)$ is holomorphic at a_ν for every $\nu \in \mathbb{N}$.

Therefore Φ is an entire holomorphic function for $z \in \mathbb{C}$, and

$$\Phi(a_\nu) = c_{\nu,0}, \quad \Phi'(a_\nu) = c_{\nu,1}, \quad \Phi''(a_\nu) = c_{\nu,2}, \dots, \Phi^{(\delta_\nu)}(a_\nu) = c_{\nu,\delta_\nu} \tag{2.12}$$

for all $\nu \in \mathbb{N}$, as required. □

Remark 2.1. The entire function Φ is not unique, as satisfying (2.3), since we can always supplement the point set $\{a_\nu \mid \nu \in \mathbb{N}\}$ by extra points - with arbitrarily assigned values for Φ at these new points of $\mathbb{C}$. In particular, if the initially prescribed point set is finite, it can be supplemented by an infinite set of points so that the augmented point set has no accumulation points in $\mathbb{C}$, and the corresponding construction can then be completed.

We should note further that no restrictions are being imposed on the growth rate of $\Phi(z)$. Moreover, in the theorems of Weierstrass and Mittag-Leffler the functions H and M, respectively, are given through explicit formulas, and accordingly we can consider Φ as explicitly constructed, compare [13].

We shall use these results to construct an entire function Φ_r, for each real $r > 0$, with specified values for Φ_r, and for a finite number of derivatives of Φ_r, at each point of the finite set $a_n = n$ for $n \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$, that is, at the origin and the integral points on the positive x-axis in the complex plane.

Definition 2.1. *For each fixed real number $r > 0$ we select an entire holomorphic function $\Phi_r(z)$ for $z \in \mathbb{C}$, with the specified values for each integer $\delta \geq 0$, as designated by:*

$$
\begin{array}{rlll}
\Phi_r(n) = e^{(e^{rn})} & at & n = 0, 1, 2, 3, 4, \dots \\
\Phi_r'(n) = \mathrm{Exp}_1(e^{(e^{rn})}) & at & n = 1, 2, 3, 4, \dots \\
\Phi_r''(n) = \mathrm{Exp}_2(e^{(e^{rn})}) & at & n = 2, 3, 4, \dots \\
\vdots & & \\
\Phi_r^{(\delta)}(n) = \mathrm{Exp}_\delta(e^{(e^{rn})}) & at & n = \delta, \delta+1, \delta+2, \dots \\
\vdots & & \\
etc & &
\end{array}
\tag{2.13}
$$

Notation 2.1. For each fixed order $\delta \geq 0$ of differentiation for Φ_r, the value $\Phi_r^{(\delta)}$ is prescribed at each integral point $n \geq \delta$ on the real axis of the complex plane. Namely,

$$\Phi_r^{(\delta)}(n) = \mathrm{Exp}_{\delta+2}(rn) = \mathrm{Exp}_\delta(e^{(e^{rn})}). \tag{2.14}$$

Here we use the notation for composite multi-exponentials:

$$
\begin{aligned}
\mathrm{Exp}_0(z) &:= z \\
\mathrm{Exp}_1(z) &:= \mathrm{Exp}(z) = e^z \\
\mathrm{Exp}_2(z) &:= \mathrm{Exp}(\mathrm{Exp}_1(z)) = e^{(e^z)} \\
&\;\vdots \\
\mathrm{Exp}_{\delta+1}(z) &:= \mathrm{Exp}(\mathrm{Exp}_\delta(z)) = \mathrm{Exp}_\delta(e^z).
\end{aligned}
\tag{2.15}
$$

Hence each $\mathrm{Exp}_\delta(z)$ is an entire holomorphic function in $\mathbb{C}$, and, for real $x \geq 0$, $\mathrm{Exp}_\delta(x)$ is monotone increasing in $x > 0$, and furthermore

$$\mathrm{Exp}_{\delta+1}(x) > \mathrm{Exp}_\delta(x). \tag{2.16}$$

Moreover, for each fixed $\delta \geq 0$, $\mathrm{Exp}_\delta(rx)$ is monotone increasing in r for each fixed $x > 0$, and in x for each fixed $r > 0$. Also for fixed $\delta \geq 1$, when $x > 0$

$$\frac{d}{dx}\mathrm{Exp}_\delta(x) = \mathrm{Exp}_\delta(x)\frac{d}{dx}(\mathrm{Exp}_{\delta-1}(x)) > 1. \tag{2.17}$$

Remark 2.2. We emphasize that in the specifications (2.13), at each fixed integral point $z = n \geq 0$, only a finite set of numerical values for Φ_r and its derivatives is prescribed. For instance, we observe that:

(2.18) at $n = 0$ specify $\Phi_r(0) = e$

at $n = 1$ specify $\Phi_r(1) = \mathrm{Exp}_2(r), \Phi_r'(1) = \mathrm{Exp}_3(r)$

at $n = 2$ specify $\Phi_r(2) = \mathrm{Exp}_2(2r), \Phi_r'(2) = \mathrm{Exp}_3(2r), \Phi_r''(2) = \mathrm{Exp}_4(2r)$

$\vdots$

and at $n = \delta$ we specify

$$\Phi_r(\delta) = \mathrm{Exp}_2(\delta r), \Phi_r'(\delta) = \mathrm{Exp}_3(\delta r), \dots, \Phi_r^{(\delta)}(\delta) = \mathrm{Exp}_{\delta+2}(\delta r).$$

Under these circumstances the existence Theorem 2.1 applies, and henceforth we consider the family of entire functions $\{\Phi_r \mid r > 0\}$ to be well-defined.

Our next goal is to show that the family $\{\Phi_r \mid r > 0\}$ is diff-independent (over $\mathbb{C}$), within the differential field $\mathbb{M}$. This program will be accomplished in several steps: The guiding idea is to examine the respective growth rates of Φ_r and its derivatives, for each $r > 0$, along the sequence of integral points $x = n \geq 0$, and to use Definition 2.1 to demonstrate that there cannot exist a (non-trivial) polynomial identity compatible with the prescribed data (2.13).

Before commencing this technical analysis it seems sensible to offer some explanation for the startlingly peculiar Definition 2.1 where $\Phi_r(n) = \mathrm{Exp}_2(rn)$. While for a single fixed value of r it might appear simpler to try the value $\mathrm{Exp}_1(rn)$ for $\Phi_r(n)$, this approach introduces confusion when several values $r_1 < r_2 < \cdots < r_\ell$ are involved in the polynomial algebra (e.g. $(e^{r_1 n})^2 = e^{r_2 n}$, say for $r_1 = 1, r_2 = 2$).

Notation 2.2. Let F_1 and F_2 be complex-valued functions defined for all integers $n > N$, for some choice of $N \in \mathbb{N}$, and assume that F_2 is non-vanishing. Then

$$F_2(n) >> F_1(n) \tag{2.19}$$

shall mean

$$\lim_{n\to\infty} \frac{F_1(n)}{F_2(n)} = 0. \tag{2.20}$$

The next two lemmas give useful estimates on the comparative rates of growth for the various multi-exponentials arising in Definition 2.1.

Lemma 2.1. *Fix a real number $r_2 > 0$ and an integer $\delta \geq 2$, as before. Then, for all positive real numbers $r_1 < r_2$ and $r > 0$, and for all positive integers a, b, the estimate obtains (for all large $n \in \mathbb{N}$, as in Notation 2.2 above):*

$$\mathrm{Exp}_\delta(r_2 n) >> [\mathrm{Exp}_{\delta-1}(rn)]^a [\mathrm{Exp}_\delta(r_1 n)]^b. \tag{2.21}$$

Proof. For $\delta = 2$ the conclusion of the lemma asserts

$$\lim_{n\to\infty} \frac{e^{arn} e^{(be^{r_1 n})}}{e^{e^{(r_2 n)}}} = \lim_{n\to\infty} e^{arn} e^{(be^{r_1 n} - e^{r_2 n})} = 0. \tag{2.22}$$

But (2.22) is valid because, using natural logarithms, we observe that

$$\ln e^{arn} e^{(be^{r_1 n} - e^{r_2 n})} = arn + [be^{r_1 n} - e^{r_2 n}],$$

which tends towards $-\infty$ as $n \to \infty$.

Now proceed by mathematical induction for $\delta \geq 2$, starting with the known result (2.22) for $\delta = 2$. Thus fix some integer $\delta \geq 2$ and assume that the result (2.21) is valid for all integers in the interval $[2, \delta]$. We must then verify that

$$\text{Exp}_{\delta+1}(r_2 n) >> [\text{Exp}_\delta(rn)]^a [\text{Exp}_{\delta+1}(r_1 n)]^b. \tag{2.23}$$

But the induction hypothesis (2.21) implies that

$$\lim_{n\to\infty} \frac{\text{Exp}_{\delta-1}(rn)}{\text{Exp}_\delta(r_2 n)} = 0 \text{ and } \lim_{n\to\infty} \frac{\text{Exp}_\delta(r_1 n)}{\text{Exp}_\delta(r_2 n)} = 0. \tag{2.24}$$

Hence, for all sufficiently large $n \in \mathbb{N}$,

$$\text{Exp}_{\delta-1}(rn) < \frac{1}{3a} \text{Exp}_\delta(r_2 n) \text{ and } \text{Exp}_\delta(r_1 n) < \frac{1}{3b} \text{Exp}_\delta(r_2 n), \tag{2.25}$$

and thus

$$[a \,\text{Exp}_{\delta-1}(rn) + b \,\text{Exp}_\delta(r_1 n)] - \text{Exp}_\delta(r_2 n) < -\frac{1}{3} \text{Exp}_\delta(r_2 n).$$

Therefore

$$\lim_{n\to\infty} [a \,\text{Exp}_{\delta-1}(rn) + b \,\text{Exp}_\delta(r_1 n)] - \text{Exp}_\delta(r_2 n) = -\infty, \tag{2.26}$$

which is equivalent to the required conclusion (2.23), and the induction argument is complete. □

Lemma 2.2. *Fix a real number $r_2 > 0$ and an integer $\delta \geq 2$, as before. Consider any finite (or empty) sets of positive real numbers $\rho_1, \rho_2, \ldots \rho_t < r_2$, also positive real numbers $\sigma_1, \sigma_2, \ldots, \sigma_s$ and integers $1 \leq \delta_1, \delta_2, \ldots, \delta_s < \delta$.*

Then for each complex polynomial Q, in $1 + s + t$ indeterminates, the estimate obtains (for all large $n \in \mathbb{N}$, as in Notation 2.2 above):

$$\text{Exp}_\delta(r_2 n) >> Q(n, \text{Exp}_{\delta_1}(\sigma_1 n), \ldots, \text{Exp}_{\delta_s}(\sigma_s n), \text{Exp}_\delta(\rho_1 n), \ldots, \text{Exp}_\delta(\rho_t n)). \tag{2.27}$$

Proof. We present the proof for the case $s \geq 1$, $t \geq 1$ and omit the details for the simpler cases where $s = 0$ or $t = 0$, that is, where some of the indicated indeterminates are absent.

Because of the monotonicity of $\text{Exp}_\ell(rn)$ in $\ell \geq 1$ (for each fixed $r > 0$), and also in $r > 0$ (for each fixed $\ell \geq 1$), we can set

$$r := \max\{\sigma_1, \sigma_2, \ldots, \sigma_s\} + 1 \tag{2.28}$$

and note that for all $n \in \mathbb{N}$,

$$\text{Exp}_{\delta_j}(\sigma_j n) < \text{Exp}_{\delta-1}(rn), \quad \text{for } j = 1, 2, \ldots, s. \tag{2.29}$$

Also we set

$$r_1 := \max\{\rho_1, \rho_2, \ldots, \rho_t\} < r_2 \tag{2.30}$$

and note that

$$\mathrm{Exp}_\delta(\rho_j n) < \mathrm{Exp}_\delta(r_1 n), \quad \text{for } j = 1, 2, \ldots, t. \tag{2.31}$$

Then, upon examining each term of the polynomial Q, and using the results of the prior Lemma 2.1, we obtain the required conclusion (2.27). □

The next Theorem 2.2 is a special case of the principal Theorem 2.3. Nevertheless we present it separately, because of its importance, and use the proof to illustrate the rather complicated arguments needed in the subsequent Lemma 2.3 and Theorem 2.3.

Theorem 2.2. *Fix a real number $r > 0$ and let Φ_r be an entire holomorphic function for $z \in \mathbb{C}$, with the given numerical value for $\Phi_r(z)$ and its derivatives, at the integral points $n = 0, 1, 2, 3, \ldots$ as specified by $\Phi_r(n), \Phi'(n), \ldots$ in (2.13) of Definition 2.1 above.*

Then Φ_r is diff-transcendental over $\mathbb{C}$, (or equally well over $\mathbb{C}(z)$, or $\mathbb{E}$, see Lemma 1.1 above).

Proof. Fix the entire function $\Phi_r(z)$, for $z \in \mathbb{C}$, as in (2.13) of Definition 2.1 above; and let P be a non-trivial complex polynomial in the $1+\delta$ (for some $\delta \geq 0$) indeterminates $\{w^0, w^1, \ldots w^{\delta-1}, w^\delta\}$, which we denote by

$$P(w^0, w^1, \ldots, w^{\delta-1}, w^\delta). \tag{2.32}$$

We shall demonstrate that the corresponding holomorphic function

$$P(\Phi_r(z), \Phi_r'(z), \ldots, \Phi_r^{(\delta-1)}(z), \Phi_r^{(\delta)}(z)) \not\equiv 0, \text{ for } z \in \mathbb{C}, \tag{2.33}$$

that is, this holomorphic function is not identically zero in $\mathbb{C}$ (and hence not vanishing in any non-empty open subset of the complex plane $\mathbb{C}$).

We dismiss the possibility that P is a non-zero constant, since then (2.33) is trivial - and hereafter we assume that the indeterminate w^δ appears explicitly in P in (2.32). That is, we can express P as a polynomial in the single indeterminate (or unknown) w^δ, say with degree $\ell \geq 1$,

$$P(w^0, w^1, \ldots, w^\delta) = (B_\ell)(w^\delta)^\ell + (B_{\ell-1})(w^\delta)^{\ell-1} + \cdots + (B_1)(w^\delta) + (B_0), \tag{2.34}$$

where the coefficients B_j, for $j = 0, 1, \ldots, \ell$ are complex polynomials in (some of) the remaining indeterminates $\{w^0, w^1, \ldots, w^{\delta-1}\}$, with $B_\ell(w^0, w^1, \ldots, w^{\delta-1})$ non-trivial, that is $B_\ell \not\equiv 0$ (B_ℓ is not the zero polynomial).

In order to demonstrate the conclusion (2.33), it is sufficient to prove the stronger result:

Proposition (A): *For the non-constant polynomial $P(w^0, w^1, \dots, w^\delta)$ of (2.32) define the corresponding function $\widehat{P} : \mathbb{N} \to \mathbb{C}$ obtained by the evaluation (2.33) at the integral points $n \in \mathbb{N}$ as in (2.13) (at least for $n \geq \delta$):*

$$\widehat{P}(n) := P(\mathrm{Exp}_2(rn), \mathrm{Exp}_3(rn), \dots, \mathrm{Exp}_{\delta+2}(rn)). \tag{2.35}$$

Then

$$\lim_{n\to\infty} |\widehat{P}(n)| = \infty. \tag{2.36}$$

We first prove Proposition (A) for the case where $\delta = 0$, so

$$P(w^0) = (B_\ell)[w^0]^\ell + (B_{\ell-1})[w^0]^{\ell-1} + \cdots + (B_1)[w^0] + (B_0), \tag{2.37}$$

where the complex coefficients B_j, for $j = 0, 1, \dots, \ell \geq 1$, are constants and $B_\ell \neq 0$. Now consider, as in (2.35) and (2.37),

$$\widehat{P}(n) = (B_\ell)[\mathrm{Exp}_2(rn)]^\ell + (B_{\ell-1})[\mathrm{Exp}_2(rn)]^{\ell-1} + \cdots + (B_1)[\mathrm{Exp}_2(rn)] + (B_0). \tag{2.38}$$

Because $\lim_{n\to\infty} \mathrm{Exp}_2(rn) = \infty$, it is trivial that

$$\mathrm{Exp}_2(rn) >> B_j \quad , \quad \text{for each } j = 0, 1, \dots, \ell, \tag{2.39}$$

and then

$$[\mathrm{Exp}_2(rn)]^\ell >> (B_{\ell-1})[\mathrm{Exp}_2(rn)]^{\ell-1} + \cdots + (B_1)[\mathrm{Exp}(rn)] + (B_0). \tag{2.40}$$

Then (2.36) holds, and consequently Proposition (A) is verified for the case $\delta = 0$.

Next examine the case $\delta = 1$, so

$$P(w^0, w^1) = B_\ell(w^0)[w^1]^\ell + B_{\ell-1}(w_0)[w^1]^{\ell-1} + \cdots + B_1(w^0)[w^1] + B_0(w^0), \tag{2.41}$$

where the coefficients $B_j(w^0)$, for $j = 0, 1, \dots, \ell \geq 1$, are complex polynomials in the single indeterminate w^0, and furthermore $B_\ell(w^0)$ is non-trivial. As before, write (at least for $n \geq 1$)

$$\widehat{P}(n) = P(\Phi_r(n), \Phi_r'(n)) = \widehat{B}_\ell(n)[\mathrm{Exp}_3(rn)]^\ell + \cdots + \widehat{B}_1(n)[\mathrm{Exp}_3(rn)] + \widehat{B}_0(n), \tag{2.42}$$

where, using the evaluations (2.13) analogously to (2.35), we denote, for $n \in \mathbb{N}$,

$$\widehat{B}_j(n) := B_j(\mathrm{Exp}_2(rn)) \quad , \quad \text{for } \; j = 0, 1, \dots, \ell. \tag{2.43}$$

Since $B_\ell(w^0)$ is non-trivial, either

(i) B_ℓ is a non-zero constant, or
(ii) $B_\ell(w^0)$ involves the single indeterminate w^0 explicitly, as analogous to $P(w_0)$ of (2.37) and hence $\lim_{n\to\infty} |\widehat{B}_\ell(n)| = \infty$ (compare (2.36)).

In either case (i) or (ii) it is immediate that

$$\liminf_{n\to\infty} |\widehat{B}(n)| > 0. \tag{2.44}$$

Now use Lemmas 2.1 and 2.2 above to conclude that

$$\mathrm{Exp}_3(rn) >> \widehat{B}_j(n) \quad \text{for } j = 0, 1, \ldots, \ell, \tag{2.45}$$

so

$$[\mathrm{Exp}_3(rn)]^\ell >> \widehat{B}_{\ell-1}(n)[\mathrm{Exp}_3(rn)]^{\ell-1} + \cdots + \widehat{B}_1(n)[\mathrm{Exp}_3(rn)] + \widehat{B}_0(n). \tag{2.46}$$

Thus (2.46), combined with (2.44), show that

$$\lim_{n\to\infty} |\widehat{P}(n)| = \infty,$$

so the assertion (2.36) of Proposition (A) holds for the case $\delta = 1$.

It is now clear how to proceed with the mathematical induction on the parameter $\delta \geq 2$, in order to prove Proposition (A) in all cases. Thus fix an integer $\Delta \geq 2$ and assume the induction hypothesis that (2.36) holds for each positive integer $\delta < \Delta$. Under these circumstances consider a non-constant complex polynomial in the indeterminates $\{w^0, w^1, \ldots, w^{\Delta-1}, w^\Delta\}$, so the polynomial

$$P(w^0, w^1, \ldots, w^{\Delta-1}, w^\Delta) \tag{2.47}$$

involves explicitly the indeterminate w^Δ. Then P in (2.47) can be expressed as a polynomial in the single indeterminate w^Δ, so

$$P(w^0, w^1, \ldots, w^{\Delta-1}, w^\Delta) = (B_\ell)[w^\Delta]^\ell + (B_{\ell-1})[w^\Delta]^{\ell-1} + \cdots + (B_1)[w^\Delta] + (B_0), \tag{2.48}$$

where the coefficients B_j, for $j = 0, 1, \ldots, \ell \geq 1$, are complex polynomials in some of the remaining indeterminates $\{w^0, w^1, \ldots, w^{\Delta-1}\}$ and further $B_\ell(w^0, w^1, \ldots, w^{\delta'})$, say for $\delta' \leq \Delta - 1$, is non-trivial and thus subject to the induction hypothesis, provided B_ℓ is non-constant. Hence we deduce

$$\liminf_{n\to\infty} |\widehat{B}_\ell(n)| > 0\ . \tag{2.49}$$

As before, define for $n \in \mathbb{N}$,

$$\widehat{P}(N) := P(\mathrm{Exp}_2(rn), \mathrm{Exp}_3(rn), \ldots \mathrm{Exp}_{\Delta+2}(rn))\ , \tag{2.50}$$

(agreeing with the evaluation (2.33) for all $n \geq \Delta$), and similarly just as in (2.43),

$$\widehat{B}_j(n) := B_j(\mathrm{Exp}_2(rn), \mathrm{Exp}_3(rn), \ldots, \mathrm{Exp}_{\Delta+1}(rn)) \quad , \text{ for } j = 0, 1, \ldots, \ell. \tag{2.51}$$

Then using Lemmas 2.1 and 2.2 as before, we conclude

$$[\mathrm{Exp}_{\Delta+2}(rn)]^\ell >> \widehat{B}_{\ell-1}(n)[\mathrm{Exp}_{\Delta+2}(rn)]^{\ell-1} + \cdots + \widehat{B}_1(n)[\mathrm{Exp}_{\Delta+2}(rn)] + \widehat{B}_0(n). \tag{2.52}$$

Therefore (2.36) of Proposition (A) holds for $\widehat{P}(n)$, corresponding to P in (2.48), and the induction is completed to prove Theorem 2.2. $\square$

The induction mechanisms of Theorem 2.2 become disconcertingly complicated when applied to the next Lemma 2.3. In order to simplify these difficulties we shall introduce some special notation to clarify the arguments.

For each non-empty, finite, ordered set of non-negative integers, say $\delta_1, \delta_2, \ldots, \delta_N \geq 0$ (without any assumptions on their relative magnitudes, and all possible $N \in \mathbb{N}$ allowed), consider polynomials, with complex coefficients, say

$$P(w_1^0, w_1^2, \ldots, w_1^{\delta_1}, w_2^0, w_2^1, \ldots, w_2^{\delta_2}, \ldots, w_N^0, w_N^1, \ldots, w_N^{\delta_N}) \tag{2.53}$$

in $\sum_{j=1}^{N}(1 + \delta_j)$ indeterminates (or unknowns), arranged as indicated (i.e., P can be interpreted as a "*differential polynomial*" with N "*differential indeterminates*" $\{w_1^0, w_2^0, \ldots, w_N^0\}$ whose derivatives of highest order $\{\delta_1, \delta_2, \ldots, \delta_N\}$, respectively, appear explicitly in P see Remark 1.2 above). Thus we assume that for each fixed $N \in \mathbb{N}$, each of the indeterminates $w_1^{\delta_1}, w_2^{\delta_2}, w_N^{\delta_N}$ appears explicitly in P - that is, each of these N indeterminates occurs in some term of P having a non-zero coefficient. In particular, we note that each such polynomial P is non-trivial, in fact, is non-constant.

We now denote the set of all such non-constant polynomials, for specified integers $\{\delta_1, \delta_2, \ldots, \delta_N\}$, by

$$\mathcal{P}\{\delta_1, \delta_2, \ldots, \delta_N\}, \tag{2.54}$$

and the totality of all these sets, for each fixed $N \in \mathbb{N}$ by

$$\mathcal{P}_N := \bigcup_{\{\delta_1, \ldots, \delta_N\}} \mathcal{P}\{\delta_1, \delta_2, \ldots, \delta_N\}. \tag{2.55}$$

In addition we set

$$\mathcal{P} = \bigcup_{N \in \mathbb{N}} \mathcal{P}_N. \tag{2.56}$$

Further, for each polynomial $P \in \mathcal{P}_N$, the height of P is defined by

$$H(P) := \max\{\delta_1, \delta_2, \ldots, \delta_N\}, \tag{2.57}$$

and the grade of P is

$$G(P) := \text{card } \{\delta_j = H(P) \mid j = 1, 2, \ldots, N\}.$$

In other words, $H(P)$ is the greatest δ_j, for $j = 1, 2, \ldots, N$, and $G(P)$ is the number of these δ_j which equal the greatest one.

Lemma 2.3. *Let $P \in \mathcal{P}_N$, for a fixed $N \in \mathbb{N}$, be a polynomial, as in (2.53)-(2.55) say for $\{\delta_1, \delta_2, \ldots, \delta_N\}$:*

$$P(w_1^0, w_1^1, \ldots, w_1^{\delta_1}, w_2^0, w_2^1, \ldots, w_2^{\delta_2}, \ldots, w_N^0, w_N^1, \ldots, w_N^{\delta_N}). \tag{2.58}$$

Then for each ordered set $\{r_1, r_2, \dots, r_N\}$ *of distinct positive real numbers, define the corresponding function*

$$\widehat{P} : n \to \widehat{P}(n) \quad \textit{for } \mathbb{N} \to \mathbb{C}, \textit{ by} \tag{2.59}$$

$$\begin{aligned}\widehat{P}(n) := P(&\mathrm{Exp}_2(r_1 n), \mathrm{Exp}_3(r_1 n), \dots, \mathrm{Exp}_{\delta_1+2}(r_1 n), \dots, \\ &\mathrm{Exp}_2(r_N n), \mathrm{Exp}_3(r_N n), \dots, \mathrm{Exp}_{\delta_N+2}(r_N n).\end{aligned} \tag{2.60}$$

Then

$$\lim_{n\to\infty} |\widehat{P}(n)| = \infty \tag{2.61}$$

Proof. First take $N = 1$, so $P(w_1^0, w_1^1, \dots w_1^{\delta_1})$ is a polynomial in $\mathcal{P}_1$, just as in (2.32) of Theorem 2.2 above. Then according to the proof of Theorem 2.2, especially Proposition (A) with the conclusion (2.36), the limit in (2.61) holds - whatever the choice of the positive number $\{r_1\}$.

Accordingly we now consider $N \geq 2$. Using an argument by contradiction, we suppose that there exists a non-empty set $\Sigma \subset \mathcal{P}$ such that $P \in \Sigma$ yields, for some suitable set of distinct positive numbers $\{r_1, r_2, \dots, r_N\}$, a corresponding function $\widehat{P}(n)$, as in (2.60), which does not have an infinite limit as $n \to \infty$. In other words, suppose

$$\liminf_{n\to\infty} |\widehat{P}(n)| < \infty. \tag{2.62}$$

In more detail we now make the following selections:

(a) Let $N \geq 2$ be the smallest integer for which $\mathcal{P}_N$ contains a polynomial $P \in \Sigma$. That is, P as in (2.58) together with some set $\{r_1, r_2, \dots, r_N\}$ yields $\widehat{P}(n)$ satisfying (2.62).

(b) Let $h \geq 0$ be the smallest integer for which there exists $P \in \mathcal{P}_N$, as in (a) above, with $H(P) = h$ and yet (2.62) holds.

(c) Let $\gamma \geq 1$ be the smallest integer for which there exists some $P \in \mathcal{P}_N$, satisfying (a) and (b) above, with $G(P) = \gamma$, and yet (2.62) holds.

Now fix a complex polynomial $P \in \mathcal{P}_N$ as in (2.58), with $\{r_1, r_2, \dots, r_N\}$, such that

$$H(P) = h, \ G(P) = \gamma \tag{2.63}$$

and yet (2.62) holds. With these data in mind, examine P with given $\{\delta_1, \delta_2, \dots, \delta_N\}$ and $\{r_1, r_2, \dots, r_N\}$, and define the unique integer m in $1 \leq m \leq N$ such that:

$$\delta_m = H(P) \ , \text{ and } r_m = \max\{r_j \mid \delta_j = H(P)\}. \tag{2.64}$$

That is, r_m is the greatest among the numbers $\{r_j\}$ for which the corresponding $\delta_j = H(P)$. For instance, if $G(P) = 1$, then there is a unique $\delta_m = H(P)$ and a unique corresponding positive number r_m - otherwise take m so that r_m has the largest value among the r_j for which $\delta_j = \max\{\delta_1, \delta_2, \dots, \delta_N\}$.

Now write P as a polynomial in the single indeterminate $w_m^{\delta_m}$, so

$$P = (B_\ell)[w_m^{\delta_m}]^\ell + (B_{\ell-1})[w_m^{\delta_m}]^{\ell-1} + \cdots + (B_1)[w_m^{\delta_m}] + (B_0) \tag{2.65}$$

where the coefficients B_j, for $j = 0, 1, \ldots, \ell \geq 1$, are complex polynomials in the remaining indeterminates, upon omitting $w_m^{\delta_m}$ (and possibly others). Furthermore the polynomial B_ℓ is non-trivial.

Based on Lemmas 2.1 and 2.2 above, we can conclude that

$$\mathrm{Exp}_{\delta_m+2}(r_m n) >> \widehat{B}_j(n) \quad , \text{ for } j = 0, 1, 2, \ldots, \ell. \tag{2.66}$$

As before, $\widehat{B}_j(n)$ is defined using B_j, upon the substitutions $w_k^\delta \to \mathrm{Exp}_{\delta+2}(r_k n)$, for $0 \leq \delta \leq \delta_k$, $k = 1, 2, \ldots, N$, compare (2.51). Hence, just as in the proof of Theorem 2.2:

$$\text{If } \liminf_{n\to\infty} |\widehat{B}_\ell(n)| > 0, \text{ then } \lim_{n\to\infty} |\widehat{P}(n)| = \infty. \tag{2.67}$$

The remainder of our proof rests on examining the growth of $|\widehat{B}_\ell(n)|$. From the expression (2.65) it follows that one of the following four assertions must obtain:

(i) B_ℓ is a non-zero constant,
(ii) $B_\ell \in \mathcal{P}_n$, for some $n \leq N-1$ (say upon suitably renumbering the differential invariants)
(iii) $B_\ell \in \mathcal{P}_N$ with $H(B_\ell) < h$,
(iv) $B_\ell \in \mathcal{P}_N$ with $H(B_\ell) = h$ and $G(P) < \gamma$.

Thus in any of these cases, the specifications for $P \in \mathcal{P}_N$, and the conditions (a) (b) (c) above, assert that $B_\ell(w_1^0, \ldots, w_N^{\delta_N})$, with any selection of N distinct positive numbers $\{r_j\}$, must yield a corresponding function $\widehat{B}_\ell(n)$ which is either a constant or else

$$\lim_{n\to\infty} |\widehat{B}_\ell(n)| = \infty. \tag{2.68}$$

In any case

$$\liminf_{n\to\infty} |\widehat{B}_\ell(n)| > 0, \tag{2.69}$$

and thus

$$\lim_{n\to\infty} |\widehat{P}(n)| = \infty. \tag{2.70}$$

But (2.70) contradicts the defining property (2.62) for P and $\widehat{P}(n)$. Thus our initial supposition at the start of the proof of this lemma, that $\Sigma \subset \mathcal{P}$ is non-empty, is contradicted.

Therefore we conclude that $\Sigma \subset \mathcal{P}$ is indeed empty, and every polynomial $P \in \mathcal{P}_N$ must satisfy the required conclusion (2.61) - and this is valid for every $N \in \mathbb{N}$. □

We can now assert and demonstrate our principal theorem.

Theorem 2.3. *The family of entire functions (see Definition 2.1 and Remark 2.2 above)*

$$\{\Phi_r \mid r > 0\} \tag{2.71}$$

is diff-independent over $\mathbb{C}$ *(or, equally well, over* $\mathbb{C}(z)$ *or* $\mathbb{E}$*). Therefore the differential transcendence degree of* $\mathbb{M}$ *over* $\mathbb{C}$ *(or, equally well, over* $\mathbb{C}(z)$ *or* $\mathbb{E}$*, see Theorem 1.2) is the cardinality of the continuum:*

$$\textit{Diff-Trans}\ \partial^0\mathbb{M}/\mathbb{C} = \mathfrak{c}. \tag{2.72}$$

Proof. By Lemma 2.3 each finite subfamily

$$\{\Phi_{r_1}, \Phi_{r_2}, \ldots, \Phi_{r_N}\} \text{ for distinct positive numbers } \{r_1, r_2, \ldots, r_N\}, \tag{2.73}$$

is diff-independent over $\mathbb{C}$ (and hence over $\mathbb{C}(z)$ or $\mathbb{E}$, see Lemma 1.1 in Section 1 above). Thus the family (2.71) is diff-independent over $\mathbb{C}$, and can be augmented to form a diff-transcendence basis for $\mathbb{M}$ over $\mathbb{C}$. Therefore

$$\text{Diff-Trans}\ \partial^0\ \mathbb{M}/\mathbb{C} \geq \ \text{card}\ \{\Phi_r | r > 0\} = \mathfrak{c}. \tag{2.74}$$

However, $\mathbb{M}$ is the quotient field of the ring of all entire holomorphic functions in $\mathbb{C}$ (each of which is defined by a complex power series). This demonstrate that

$$\text{card}\ \mathbb{M} = \mathfrak{c}. \tag{2.75}$$

Hence

$$\text{Diff-Trans}\ \partial^0\mathbb{M}/\mathbb{C} = \mathfrak{c},$$

as asserted in the theorem. □

As a concluding remark we observe that there is no claim that the family $\{\Phi_r\ |r > 0\}$ constitutes a diff-transcendence basis for $\mathbb{M}$ over $\mathbb{C}$; the Theorem 2.3 merely asserts that it is part of such a basis.

Appendix A. The Gamma and Zeta Functions

In 1887 O. Hölder proved [4] that the Euler gamma function, $\Gamma \in \mathbb{M}$, where

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt \qquad \text{for Re } z > 0, \tag{A.1}$$

satisfies no polynomial (algebraic) ordinary differential equation with coefficients in $\mathbb{C}(z)$. In the terminology of this current paper:

Theorem A.1. *(Hölder) The Euler gamma function* $\Gamma \in \mathbb{M}$ *is diff-transcendental over* $\mathbb{C}(z)$*. (or, equally well, over* $\mathbb{C}$ *or* $\mathbb{E}$*, as in Lemma 1.1 above).*

In this Appendix we extend the methods of Hölder to apply to the Riemann zeta function, $\zeta \in \mathbb{M}$, where

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad \text{for Re } z > 1, \tag{A.2}$$

and then to several related meromorphic functions.

Lemma A.1. *Let $\Phi \in \mathbb{M}$ be diff-transcendental over $\mathbb{C}$ (or, equally well, over $\mathbb{C}(z)$ or $\mathbb{E}$). Then the functions*

$$\Psi(z) := \Phi(az + b) \quad , \quad a \neq 0 \text{ and } b \in \mathbb{C}, \tag{A.3}$$

and

$$\chi(z) := \Phi(\sin az) \quad , \quad a \neq 0 \text{ in } \mathbb{C}, \tag{A.4}$$

are each in $\mathbb{M}$, and further each is diff-transcendental over $\mathbb{C}$.

Proof. Since $\Phi \in \mathbb{M}$ is the quotient of two entire holomorphic functions, so are Ψ and χ; which shows that both Ψ and $\chi \in \mathbb{M}$.

Suppose Ψ is diff-algebraic over $\mathbb{C}$, so then there exists a non-trivial polynomial expression

$$P(w^0, w^1, \ldots, w^\delta) \tag{A.5}$$

with complex coefficients, in $1 + \delta \geq 1$ indeterminates (say, with w^δ appearing explicitly in P), such that the meromorphic function

$$P(\Psi(z), \Psi'(z), \ldots, \Psi^{(\delta)}(z) = 0 \ , \quad \text{for all } z \in \mathbb{C}. \tag{A.6}$$

But in this case

$$P(\Phi(az + b), a\Phi'(az + b), \ldots, a^\delta \Phi^\delta(az + b)) \equiv 0 \tag{A.7}$$

and hence, for all $z \in \mathbb{C}$,

$$P(\Phi(z), a\Phi'(z), \ldots, a^\delta \Phi^{(\delta)}(z)) \equiv 0. \tag{A.8}$$

Since $a \neq 0$, the assertion in (A.8) yields a non-trivial polynomial differential equation with Φ as a solution, which is impossible since Φ is diff-transcendental over $\mathbb{C}$. This contradiction shows that $\Psi \in \mathbb{M}$ must be diff-transcendental over $\mathbb{C}$.

Next consider the meromorphic function χ, where

$$\chi(z) = \Phi(\sin az) \ , \ \text{for } z \in \mathbb{C}.$$

As before, suppose that χ is diff-algebraic over $\mathbb{C}$, so there then exists a non-trivial polynomial expression, say $P(w^0, w^1, \ldots, w^\delta)$ as in (A.5), with

$$P(\chi(z), \chi'(z), \ldots, \chi^{(\delta)}(z)) \equiv 0, \ \text{for } z \in \mathbb{C}. \tag{A.9}$$

In this case, for each $z \in \mathbb{C}$,

$$P(\Phi(\sin az),\ a\ \cos az\ \Phi'(\sin az), \ldots, \frac{d^\delta}{dz^\delta}\Phi(\sin az)) \equiv 0. \tag{A.10}$$

Now define the holomorphic map (near $z = 0, t = 0$)

$$z \to t := \sin az\,, \tag{A.11}$$

by which some small disk $B_\varepsilon = \{t\ |t| < \varepsilon\}$, $\varepsilon > 0$, in the complex t-plane is the topological image of an open neighborhood U of $z = 0$ in the complex z-plane. Thus for each number $t \in B_\varepsilon$ there exists a unique number $z \in U$ such that $\sin az = t$, and we write $z = a^{-1}\arcsin t$. Moreover

$$\cos az = \sqrt{1-t^2}, \quad \text{with } \operatorname{Re}\sqrt{1-t^2} > 0, \tag{A.12}$$

and we note that $\sqrt{1-t^2}$ is then a meromorphic function in B_ε, so $\sqrt{1-t^2} \in \mathbb{M}_\varepsilon$, where $\mathbb{M}_\varepsilon$ is the differential field of all meromorphic functions in B_ε.

With this notation we can re-write (A.10) as a polynomial differential equation

$$P(\Phi(t), a\sqrt{1-t^2}\ \Phi'(t), \ldots) \equiv 0, \text{ for } t \in B_\varepsilon, \tag{A.13}$$

where the left side of (A.13) is a polynomial in $\{\Phi(t), \Phi'(t), \ldots, \Phi^{(\delta)}(t)\}$ with coefficients in the differential field $\mathbb{C} < t, \sqrt{1-t^2} > \subset \mathbb{M}_\varepsilon$. Further, the coefficient of $\Phi^{(\delta)}(t)$ is $a^\delta(1-t^2)^{\delta/2} \neq 0$, so we conclude that Φ is diff-algebraic over $\mathbb{C}\ \langle t, \sqrt{1-t^2}\rangle \subset \mathbb{M}_\varepsilon$. But $\mathbb{C}\langle t, \sqrt{1-t^2}\rangle$ is diff-algebraic over $\mathbb{C} \subset \mathbb{M}_\varepsilon$, so we conclude that $\Phi(t)$ is a solution of some polynomial differential equation

$$P_\varepsilon(\Phi(t), \Phi'(t), \ldots, \Phi^{(\delta')}(t)) \equiv 0\ , \quad \text{for } t \in B_\varepsilon, \tag{A.14}$$

for some integer $\delta' \geq 0$, where P_ε has constant coefficients. However Φ is meromorphic in the entire complex plane, so

$$P_\varepsilon(\Phi(t), \Phi'(t), \ldots, \Phi^{(\delta)}(t)) \equiv 0\ , \quad \text{for } t \in \mathbb{C}, \tag{A.15}$$

which contradicts the hypothesis that Φ is diff-transcendental over $\mathbb{C}$.

Therefore we have proved that χ is diff-transcendental over $\mathbb{C}$, as required. □

Theorem A.2. *The Riemann zeta function, $\zeta \in \mathbb{M}$, is diff-transcendental over $\mathbb{C}$ (and, equally well, over $\mathbb{C}(z)$ and $\mathbb{E}$).*

Proof. The Γ-function (A.1) and the ζ-function (A.2) are related by the functional equation [14][19],

$$\zeta(1-z) = 2(2\pi)^{-z}\cos\frac{\pi z}{2}\Gamma(z)\zeta(z), \qquad z \in \mathbb{C}. \tag{A.16}$$

Suppose, contrary to the theorem, that $\zeta(z)$ is diff-algebraic over $\mathbb{C}$. Then, by Lemma A.1, $\zeta(1-z)$ is also diff-algebraic over $\mathbb{C}$, and hence over $\mathbb{C}(z)$ and $\mathbb{E}$.

Since $2e^{-z\ln 2\pi}\cos\frac{\pi z}{2} \in \mathbb{E}$, the functional equation (A.16) implies that $\Gamma(z)$ is diff-algebraic over $\mathbb{E}$, which contradicts Hölder's Theorem A.1 above. Therefore, we conclude that $\zeta(z)$ must be diff-transcendental over $\mathbb{C}$. $\square$

In the discussions leading up to the next theorem we follow closely the methods of Hölder. We shall be concerned with a pair of meromorphic functions $\{\chi(z), \Gamma(z)\}$, each diff-transcendental over $\mathbb{C}$, and satisfying

$$\chi(z+1) \equiv \chi(z) \quad \text{and} \quad \Gamma(z+1) = z\Gamma(z); \tag{A.17}$$

for instance take $\chi(z) = \zeta(\sin 2\pi z)$ and the Euler Γ-function. Theorem A.3 asserts that $\{\chi(z), \Gamma(z)\}$ are then diff-independent over $\mathbb{C}$, that is,

$$\text{Diff-Trans } \partial^0\ \mathbb{C} < \chi\ ,\ \Gamma > /\mathbb{C} = 2 \tag{A.18}$$

or

$$\text{Diff-Trans } \partial^0\ \mathbb{C} < z, \chi, \Gamma > /\mathbb{C} < z >= 2. \tag{A.19}$$

Suppose, to the contrary, that there exists a polynomial expression (non-trivial)

$$P(v^0, v^1, \ldots, v^m, w^0, w^1, \ldots, w^n, z), \tag{A.20}$$

with complex coefficients, involving $(1+m)+(1+n)+1$ indeterminates (for various integers $m \geq 0, n \geq 0$ for different polynomials), such that for all $z \in \mathbb{C}$

$$P(\chi(z), \chi'(z), \ldots, \chi^{(m)}(z), \Gamma(z), \Gamma'(z), \ldots, \Gamma^{(n)}(z), z) \equiv 0. \tag{A.21}$$

It is no restriction to assume that both (v^m) and (w^n) appear explicitly in P in (A.21), since each of χ and Γ is individually diff-transcendental over $\mathbb{C}(z)$.

Now denote by $\mathfrak{M}$ the set of all such non-trivial polynomial expressions P in (A.20) (for all possible choices of the integers $m \geq 0, n \geq 0$), for which the corresponding meromorphic function of (A.21) vanishes in $\mathbb{C}$. The conclusion of Theorem A.3 is that this set $\mathfrak{M}$ is empty (for all choices of $m \geq 0$, $n \geq 0$), and we shall proceed by contradiction supposing that $\mathfrak{M}$ is non-empty. The first step in the proof presented below is to define a unique polynomial expression $F_0 \in \mathfrak{M}$, which is of minimal type in a certain specified sense.

Again consider a polynomial expression $P \in \mathfrak{M}$ as in (A.20), say

$$P(v^0, v^1, \ldots, v^m, w^0, w^1, \ldots, w^n, z),$$

with complex coefficients in $(1+m)+(1+n)+1$ indeterminates, and with (v^m) and (w^n) both appearing explicitly (with positive exponents) in P, and we often abbreviate this by referring to the two "differential indeterminates" (v, w): hence

$$P(v, w, z), \text{ with } v = \{v^0, v^1, \ldots, v^m\}, w = \{w^0, w^1, \ldots, w^n\}.$$

Consider $P(v, w, z)$ as a polynomial in v, w with coefficients in $\mathbb{C}(z)$, and denote the corresponding terms by (using compressed notation for $(v)^\mu(w)^\nu$, etc.)

$$A_{\mu,\nu}(z)(v)^\mu(w)^\nu = A_{\mu_0\mu_1\ldots\mu_m\nu_0\nu_1\ldots\nu_n}(z)(v^0)^{\mu_0}\ldots(v^m)^{\mu_m}(w^0)^{\nu_0}\ldots(w^n)^{\nu_n}, \tag{A.22}$$

according to the exponents (μ, ν) (non-negative integers). Then order these terms of P lexocographically by first ν_n, then $\nu_{n-1} \dots$, then ν_0; then $\mu_m, \dots$, then μ_0. That is, for exponents $(j, k) \neq (\mu, \nu)$ for two different terms, set

$$(j,k) > (\mu,\nu) \tag{A.23}$$

just in case the last non-zero number in the finite sequence

$$j_0 - \mu_0, \dots, j_m - \mu_m, k_0 - \nu_0, \dots, k_{n-1} - \nu_{n-1}, k_n - \nu_n, \tag{A.24}$$

is positive (allow zero exponents so that all $(1+m)+(1+n)$ indeterminates are recorded in both terms), otherwise $(j,k) < (\mu,\nu)$ when this last non-zero number is negative.

Definition A.1. *Consider a polynomial expression as in (A.20)*

$$P(v^0, v^1, \dots, v^m, w^0, w^1; \dots, w^n, z),$$

with complex coefficients and in $(1+m)+(1+n)+1$ *indeterminates, such that* (v^m) *and* (w^n) *appear explicitly (with positive exponents) and such that (A.17) and (A.21) hold, so then*

$$P(v,w,z) \in \mathfrak{M}. \tag{A.25}$$

Each term $A_{\mu\nu}(z)(v)^\mu(w)^\nu$ *has the height* (μ,ν) *(for exponents that are non-negative integers). Now choose the unique term of* P,

$$A_{jk}(z) \quad (v)^j(w)^k \tag{A.26}$$

of maximal height (according to the ordering (A.22) (2.23)) and denote this as the leading term of P.

Then define the height of P *to be the height* (j,k) *of its leading term.*

Definition A.2. *Give functions* χ *and* Γ *as in (A.17), and consider the (non-empty) set* $\mathfrak{M}$ *of all (non-trivial) polynomial expressions* P *satisfying (A.20) and (A.21), as in Definition A.1 above. Then there exists a unique minimal height, say* $(\widehat{j},\widehat{k})$ *among all polynomial expressions in* $\mathfrak{M}$*, and we define the set*

$$\widehat{\mathfrak{M}} \subset \mathfrak{M}$$

$$\widehat{\mathfrak{M}} := \{P(v,w,z) \in \mathfrak{M} | \quad \textit{height } P = (\widehat{j},\widehat{k})\}. \tag{A.27}$$

That is, the leading term of each $P \in \widehat{\mathfrak{M}}$ *is the minimal possible height* $(\widehat{j},\widehat{k})$ *of leading terms of polynomials in* $\mathfrak{M}$*. Since* $(\widehat{j},\widehat{k}) = (\widehat{j_0}\widehat{j_1}\dots\widehat{j_{\widehat{m}}}, \widehat{k_0}\widehat{k_1}\dots\widehat{k_{\widehat{n}}})$*, the integers* $\widehat{m} \geq 0, \widehat{n} \geq 0$ *are also fixed for all polynomials in* $\widehat{\mathfrak{M}}$.

Further, define the subset $\mathfrak{M}_0 \subset \widehat{\mathfrak{M}}$ *as those polynomial expressions, say*

$$P(v,w,z) \in \widehat{\mathfrak{M}}$$

with leading terms $A_{\widehat{j},\widehat{k}}(z)(v)^{\widehat{j}}(w)^{\widehat{k}}$, *such that the degree of the coefficient* $A_{\widehat{j}\widehat{k}}(z)$, *is the minimum amongst all such polynomial expressions arising in* $\widehat{\mathfrak{M}}$.

Lemma A.2. *Give functions* χ, Γ *and examine the sets*

$$\mathfrak{M}_0 \subset \widehat{\mathfrak{M}} \subset \mathfrak{M} \qquad \textit{(non-empty)}, \tag{A.28}$$

as in Definitions A.1 and A.2 above. Then there exists a unique polynomial expression

$$F_0(u, w, z) \in \mathfrak{M}_0, \tag{A.29}$$

whose leading term, say for simplicity of notation,

$$A_0(z)(v)^{\widehat{j}}(w)^{\widehat{k}}, \tag{A.30}$$

has a monic polynomial coefficient $A_0(z)$, *(the coefficient of the highest power of* z *in* $A_0(z)$ *is 1).*

Furthermore, for each $F(v, w, z) \in \widehat{\mathfrak{M}}$, *there exists a unique polynomial* $Q(z)$ *such that*

$$F(v, w, z) = Q(z)F_0(v, w, z). \tag{A.31}$$

Proof. Assuming $\mathfrak{M}$ is non-empty, it follows directly that $\widehat{\mathfrak{M}}$ and $\mathfrak{M}_0$ are each non-empty, since the required minima are over well-ordered sets. Hence, under these conditions, there exists some

$$F_0(v, w, z) = A_0(z)(v)^{\widehat{j}}(w)^{\widehat{k}} + \dots \quad \in \mathfrak{M}_0, \tag{A.32}$$

with $A_0(z)$ a monic polynomial of minimal degree.

Let

$$F(v, w, z) = B(z)(v)^{\widehat{j}}(w)^{\widehat{k}} + \dots \qquad \in \widehat{\mathfrak{M}}, \tag{A.33}$$

so both F_0 and F have the same height $(\widehat{j}, \widehat{k})$. Since

$$\partial^0 A_0(z) \le \partial^0 B(z), \tag{A.34}$$

we use the division algorithm to obtain

$$B(z) = Q(z)A_0(z) + R(z), \tag{A.35}$$

where $Q(z)$ is the unique quotient polynomial, and the remainder polynomial satisfies

$$R(z) \equiv 0 \quad \text{or} \quad \partial^0 R(z) < \partial^0 A_0(z)\ . \tag{A.36}$$

Now define the polynomial expression

$$F^*(v, w, z) := F(v, w, z) - Q(z)F_0(v, w, z). \tag{A.37}$$

Then, as in (A.21) (with the usual abbreviations)

$$F^*(\chi(z), \zeta(z), z) \equiv 0, \tag{A.38}$$

and hence either $F^* \equiv 0$ is trivial, or $F^* \in \mathfrak{M}$. But

$$F^*(v, w, z) = R(z)(v)^{\widehat{j}}(w)^{\widehat{k}} + \text{ lower height terms },$$

so either $F^* \equiv 0$ is trivial, or $F^* \in \widehat{\mathfrak{M}}$. However in either case of (A.36), noting the properties of $A_0(z)$, we see that F^* cannot belong to $\widehat{\mathfrak{M}}$, and hence $F^* \equiv 0$ and so

$$F(v, w, z) = Q(z)F_0(v, w, z). \tag{A.39}$$

Finally we note the uniqueness of $F_0 \in \mathfrak{M}_0$ enforced by the monic polynomial $A_0(z)$ in the leading term $A_0(z)(v)^{\widehat{j}}(w)^{\widehat{k}}$. Because, suppose $F(v, w, z)$ of (A.33) satisfies $F \in \mathfrak{M}_0$, so

$$\partial^0 B(z) = \partial^0 A_0(z), \text{ with } B(z) \text{ monic}. \tag{A.40}$$

Then (A.35) applied to this special case demonstrates that $Q(z) \equiv 1$, so by (A.39) we find that

$$F(v, w, z) = F_0(v, w, z). \tag{A.41}$$

□

Definition A.3. *Give χ, Γ and the sets $\mathfrak{M}_0 \subset \widehat{\mathfrak{M}} \subset \mathfrak{M}$, as in Definition A.2 above. Assume that $\mathfrak{M}$ is non-empty and let $F_0(v, w, z)$ be the unique polynomial expression in $\mathfrak{M}_0$ with leading term $A_0(z)(v)^{\widehat{j}}(w)^{\widehat{k}}$, and with $\partial^0 A(z)$ a minimum, as in Lemma A.2.*

Then define $F_0(v, w, z)$ as the **minimal polynomial expression** *in $\mathfrak{M}$.*

Remark A.1. On the minimal polynomial $F_0(v, w, z)$.

(a) In the multi-exponent $(\mu, \nu) = (\mu_0, \ldots, \mu_m, \nu_0, \ldots, \nu_n)$ of (A.22) each component is a non-negative integer; however, $\mu_m \geq 1$, and $\nu_n \geq 1$, for some terms in order that χ and Γ enter explicitly into $P(u, v, z) \in \mathfrak{M}$ in (A.21), as demanded by Hölders Theorem A.1 and also Theorem A.2 above. This holds in particular for $F_0(u, v, z)$.

(b) The minimal polynomial $F_0(v, w, z)$ is prime in the ring of all polynomials in (v, w, z). That is, there is no polynomial factorization

$$F_0(v, w, z) = G(v, w, z)H(v, w, z). \tag{A.42}$$

excepting the trivial case where one of G or H is a constant in $\mathbb{C}$.

To demonstrate this assertion suppose there is a non-trivial factorization (A.42) with both $G(v, w, z)$ and $H(v, w, z)$ non-constant polynomials. Because we have $F_0(\chi(z), \Gamma(z), z) \equiv 0$ for $z \in \mathbb{C}$ (as in A.21)), we note that

$$G(\chi(z), \Gamma(z), z)H(\chi(z), \Gamma(z), z) \equiv 0 \qquad \text{for } z \in \mathbb{C}.$$

Then $G(\chi(z), \Gamma(z), z) \equiv 0$ on some closed subset of $\mathbb{C}$, and similarly for $H(\chi(z), \Gamma(z), z)$. But the complex plane $\mathbb{C}$ is not the union of two closed nowhere dense subsets, so, say

$$G(\chi(z), \Gamma(z), z) \equiv 0 \tag{A.43}$$

on some non-empty open set in $\mathbb{C}$, and therefore for all $z \in \mathbb{C}$. This implies that $G(v, w, z) \in \mathfrak{M}$.

Let the leading term of $G(v, w, z)$ be $\widetilde{A}(z)(v)^{\widetilde{j}}(w)^{\widetilde{k}}$, so the height $(\widetilde{j}, \widetilde{k}) \leq (\widehat{j}, \widehat{k})$ according to (A.42). This implies that $\widetilde{j} = \widehat{j}$ and $\widetilde{k} = \widehat{k}$, and moreover $H = H(z)$ depends on z alone. Thus $G(v, w, z) \in \widehat{\mathfrak{M}}$, and further from (A.42)

$$A_0(z) = \widetilde{A}(z)H(z) \quad . \tag{A.44}$$

But this contradicts the construction of

$$F_0(v, w, z) = A_0(z)(v)^{\widehat{j}}(w)^{\widehat{k}} + \dots ,$$

with $\partial^0 A_0(z)$ minimal within polynomials of $\widehat{M}$, as in Definition A.2. Consequently $F_0(v, w, z)$ is a prime polynomial expression, as asserted.

In particular, $F_0(v, w, z)$ is not divisible (within the corresponding polynomial ring) by any of the factors

$$v^0, v^1, \dots, v^m, w^0, w^1, \dots, w^n, \text{ or } z - \alpha \text{ for } \alpha \in \mathbb{C}. \tag{A.45}$$

Lemma A.3. *Give functions χ, Γ and the sets*

$$\mathfrak{M}_0 \subset \widehat{\mathfrak{M}} \subset \mathfrak{M} \qquad \textit{(non-empty)}$$

as in Lemma A.2. Let the unique minimal polynomial expression, as in Definition A.3, be

$$F_0(v, w, z) = A_0(z)(v^0)^{\widehat{j}_0}(v^1)^{\widehat{j}_1} \dots (v^{\widehat{m}})^{\widehat{j}_{\widehat{m}}}(w^0)^{\widehat{k}_0} \dots (w^{\widehat{n}})^{\widehat{k}_{\widehat{n}}} + \dots , \tag{A.46}$$

or in abbreviated notation

$$F_0(v, w, z) = A_0(z)v^{\widehat{j}}w^{\widehat{k}} + \dots , \tag{A.47}$$

where the two differential indeterminates are

$$v = \{v^0, v^1, \dots, v^{\widehat{m}}\} \quad , \quad w = \{w^0, w^1, \dots, w^{\widehat{n}}\} \tag{A.48}$$

and the multi-exponents are

$$\widehat{j} = (\widehat{j}_0 \widehat{j}_1 \dots \widehat{j}_{\widehat{m}}) \quad , \quad \widehat{k} = (\widehat{k}_0 \widehat{k}_1 \dots \widehat{k}_{\widehat{n}}). \tag{A.49}$$

Then $F_0(v, w, z)$ satisfies the **Fundamental Functional Identity:**

$$F_0(v, zw^0, zw^1 + w^0, \dots, zw^{\widehat{n}} + \widehat{n}w^{\widehat{n}-1}, z + 1) = z^K F_0(v, w, z) \tag{A.50}$$

where

$$K = \widehat{k}_0 + \widehat{k}_1 + \dots + \widehat{k}_{\widehat{n}}. \tag{A.51}$$

Proof. The minimal polynomial expression $F_0(v,w,z)$ is nullified by the pair of functions χ, Γ so

$$F_0(\chi(z), \chi'(z), \dots, \chi^{(m)}(z), \Gamma(z), \Gamma'(z), \dots, \Gamma^{(n)}(z), z) \equiv 0, \quad \text{for } z \in \mathbb{C}, \tag{A.52}$$

(Here we write m for $\widehat{m}$, and n for $\widehat{n}$, to simplify the notation in this proof).

Now replace z by $z+1$, and use the functional equations

$$\chi(z+1) = \chi(z) \quad , \quad \Gamma(z+1) = z\Gamma(z) \tag{A.53}$$

to obtain, for $z \in \mathbb{C}$,

$$\chi'(z+1) = \chi'(z), \chi''(z+1) = \chi''(z), \dots, \chi^{(m)}(z+1) = \chi^{(m)}(z), \tag{A.54}$$

and

$$\begin{aligned} &\Gamma'(z+1) = z\Gamma'(z) + \Gamma(z),\ \Gamma''(z+1) = z\Gamma''(z) + 2\Gamma'(z), \\ &\Gamma^{(3)}(z+1) = z\Gamma^{(3)}(z) + 3\Gamma''(z), \dots, \Gamma^{(n)}(z+1) = z\Gamma^{(n)}(z) + n\Gamma^{(n-1)}(z). \\ &(\text{note: let } u = z+1 \quad \text{so compute} \\ &\Gamma(u)\big|_{u=z+1} = z\Gamma(z), \tfrac{d\Gamma}{du}\big|_{u=z+1}\tfrac{du}{dz} = \Gamma'(z+1) = z\Gamma'(z) + \Gamma(z), \ \text{etc.}) \end{aligned} \tag{A.55}$$

Hence (A.52) now yields, upon replacing z by $z+1$,

$$F_0(\chi(z), \chi'(z), \dots, \chi^{(m)}(z), z\Gamma(z), z\Gamma'(z) + \Gamma(z), \dots, z\Gamma^{(n)}(z) + n\Gamma^{(n-1)}(z), z+1) \equiv 0. \tag{A.56}$$

But (A.56) shows that the polynomial expression

$$\widehat{F}(v,w,z) := F_0(v^0, v^1, \dots, v^m, zw^0, zw^1 + w^0, \dots, zw^n + nw^{n-1}, z+1) \tag{A.57}$$

is nullified by the pair of functions χ, Γ. Thus the non-trivial polynomial expression $\widehat{F}$ belongs to $\mathfrak{M}$,

$$\widehat{F}(v,w,z) \in \mathfrak{M}. \tag{A.58}$$

We verify that $\widehat{F}(v,w,z)$ is non-trivial, and furthermore that its height is $(\widehat{j}, \widehat{k})$, which is the same as the height of the leading term of $F_0(v,w,z)$ as is displayed in (A.46) and (A.47). This indeed will guarantee that

$$\widehat{F}(v,w,z) \in \widehat{\mathfrak{M}}. \tag{A.59}$$

The terms of $\widehat{F}(v,w,z)$ include those that arise from the leading term of $F_0(v,w,z)$, namely

$$A_0(z)(v^0)^{\widehat{j}_0} \dots (v^m)^{\widehat{j}_m}(w^0)^{\widehat{k}_0}(w^1)^{\widehat{k}_1} \dots (w^n)^{\widehat{k}_n}, \tag{A.60}$$

upon the substitution $z \to z+1$, as in (A.56)(A.57). That is, examine the terms arising in $\widehat{F}(v,w,z)$ from

$$A_0(z+1)(v^0)^{\widehat{j}_0} \dots (v^m)^{\widehat{j}_m}(zw^0)^{\widehat{k}_0}(zw^1 + w^0)^{\widehat{k}_1} \dots (zw^n + nw^{n-1})^{\widehat{k}_n}, \tag{A.61}$$

and note that each such term is a product of monomials, including one selected from each of the displayed binomial powers. We shall analyse this calculation in some detail, because it illustrates a rather general process used again later in the proof of Theorem A.3. below.

We first find the term of greatest height among those terms arising in (A.61). Now (w^n) enters into (A.61) only within the factor $(zw^n + nw^{n-1})^{\widehat{k}_n}$ and there its highest power is found in $(zw^n)^{\widehat{k}_n} = z^{\widehat{k}_n}(w^n)^{\widehat{k}_n}$. In accord with the sense of order introduced in (A.23) above, we next seek the highest power of (w^{n-1}) within

$$A_0(z+1)(v^0)^{\widehat{j}_0}\dots(v^m)^{\widehat{j}_m}(zw^0)^{\widehat{k}_0}\dots(zw^{n-1}+(n-2)w^{n-2})^{\widehat{k}_{n-1}}(zw^n)^{\widehat{k}_n},$$

which occurs only in $(zw^{n-1})^{\widehat{k}_{n-1}}$. Next keep $(zw^{n-1})^{\widehat{k}_{n-1}}(zw^n)^{\widehat{k}_n}$ and seek the highest power of (w^{n-2}) in

$$A_0(z+1)(v^0)^{\widehat{j}_0}\dots(v^m)^{\widehat{j}_m}(zw^0)\dots(zw^{n-2}+(n-3)w^{n-3})^{\widehat{k}_{n-2}}(zw^{k-1})^{\widehat{k}_{n-1}}(zw^n)^{\widehat{k}_n}.$$

Upon repeating this selection of highest powers of $w^n, w^{n-1}, w^{n-2}, \dots, w^1, w^0$ (and noting that v enters without changes at the ending of the ordering), we obtain the terms with greatest height of $\widehat{F}(v,w,z)$ that arises from (A.61), namely,

$$z^K A_0(z+1)(v^0)^{\widehat{j}_0}\dots(v^m)^{\widehat{j}_m}(w^0)^{\widehat{k}_0}(w^1)^{\widehat{k}_1}\dots(w^n)^{\widehat{k}_n}, \tag{A.62}$$

where $K = \widehat{k}_0 + \widehat{k}_1 + \widehat{k}_2 + \cdots + \widehat{k}_n$. This term has the height $(\widehat{j}, \widehat{k})$ which is the same as the height of $F_0(v,w,z)$.

However, we must also consider briefly the terms of $\widehat{F}(v,w,z)$ that arise from the other terms of lesser height in $F_0(v,w,z)$ - upon the substitution $z \to z+1$. If we consider any other (non-leading) term of $F_0(v,w,z)$, say of height $(\mu,\nu) < (\widehat{j},\widehat{k})$, and repeat the previous argument, then we are led to terms of $\widehat{F}(v,w,z)$ that are of height (μ,ν) or less. Therefore (A.62) is indeed the leading term of $\widehat{F}(v,w,z)$, so we have proved the assertion (A.59). That is, the height of $\widehat{F}(v,w,z)$ is $(\widehat{j},\widehat{k})$ and $\widehat{F}(v,w,z) \in \widehat{\mathfrak{M}}$.

By Lemma A.2 there exists a polynomial $D(z)$ such that

$$\widehat{F}(v,w,z) = D(z)F_0(v,w,z), \tag{A.63}$$

and consequently, from (A.62),

$$z^K A_0(z+1) = D(z)A_0(z). \tag{A.64}$$

Since $A_0(z)$ is monic, we find that

$$\partial^0 D(z) = K \quad \text{and } D(z) \text{ is monic.} \tag{A.65}$$

Hence $D(z)$ has the form

$$D(z) = z^K + \text{ lower order terms }. \tag{A.66}$$

In order to complete the proof of the Fundamental Functional Identity (A.50), we need only show that $D(z)$ has no factor $(z-\alpha)$ for $\alpha \neq 0$.

For this purpose we now introduce a rational transformation of variables $(v, w, z) \to (v, q, t)$ into the formula (A.63), or

(A.67) $F_0(v, zw^0, zw^1 + w^0, \ldots, zw^n + nw^{n-1}, z+1) = D(z)F_0(v, w^0, w^1, \ldots, w^n, z)$.

Define $q = (q^0, q^1, \ldots, q^n)$ and $t \in \mathbb{C}$ by

$$z+1 = t \quad , \quad z = t-1$$

$$zw^0 = q^0 \quad , \quad w^0 = \frac{q_0}{t-1} \qquad = \frac{Q_0}{t-1}$$

$$zw^1 + w^0 = q^1 \quad , \quad w^1 = \frac{q_1 - w^0}{t-1} = \frac{q^1(t-1) - q_0}{(t-1)^2} = \frac{Q_1}{(t-1)^2}$$

$$\vdots$$

$$zw^{n-1} + (n-1)w^{n-2} = q^{n-1} \quad , \quad w^{n-1} = \frac{q^{n-1} - (n-1)w^{n-2}}{t-1} = \frac{Q_{n-1}}{(t-1)^n}$$

$$zw^n + nw^{n-1} = q^n \quad , \quad w^n + \frac{q^n - nw^{n-1}}{t-1} = \frac{Q_n}{(t-1)^{n+1}}$$

where $Q_0 = q_0, Q_1 = q^1/(t-1) - q_0, \ldots, Q_n =$ polynomial in q and t. Then (A.67) becomes the identity

(A.68) $F_0(v, q^0, q^1, \ldots, q^n, t) = D(t-1)F_0(v, \frac{Q_0}{t-1}, \frac{Q_1}{(t-1)^2}, \ldots, \frac{Q_n}{(t-1)^{n+1}}, t-1)$

Note that the rational function on the right side is a polynomial in $t-1$, $\frac{Q_0}{t-1}$, $\frac{Q_1}{(t-1)^2}$, $\ldots$, $\frac{Q_n}{(t-1)^{n+1}}$, and v.

Then there exists an integer $J \geq 0$ for which

(A.69) $G(v, q^0, q^1, \ldots, q^n, t) := (t-1)^J F_0(v, \frac{Q_0}{t-1}, \frac{Q_1}{(t-1)^2}, \ldots, \frac{Q_n}{(t-1)^{n+1}}, t-1)$

is a polynomial in v, q, t. That is, we obtain the polynomial identity, from (A.68),

$$(t-1)^J F_0(v, q, t) = D(t-1)G(v, q, t).$$

We recall that $F_0(v, q, t)$ has no factor $t - \alpha$ for $\alpha \in \mathbb{C}$, see Remark A.1 above. Therefore $D(t-1)G(v, q, t)$ can be divisible (within polynomial algebra) by $t - \alpha$ only when $\alpha = 1$. This implies that $D(t-1)$ is a power of $(t-1)$, or

(A.70) $$D(t-1) = (t-1)^K \text{ so } D(z) = z^K.$$

Therefore we have proved the validity of the Fundamental Function Identity (A.67) in the required format of (A.50) and (A.51):

$$F_0(v, zw^0, zw^1 + w^0, \ldots, zw^n + nw^{n-1}, z+1) = z^K F_0(v, w, z).$$

□

We finally prove the main Theorem of this Appendix.

Theorem A.3. *Let χ and Γ be meromorphic functions in $\mathbb{M}$, and assume that each is diff-transcendental over $\mathbb{C}$ (or, equally well, over $\mathbb{C}(z)$ or $\mathbb{E}$). Further assume the functional equations hold*

(A.71) $$\chi(z+1) = \chi(z) \text{ and } \Gamma(z+1) = z\Gamma(z), \text{ for } z \in \mathbb{C}.$$

Then the pair χ, Γ are diff-independent over $\mathbb{C}$ (or, equally well, over $\mathbb{C}(z)$ or $\mathbb{E}$, see Lemma 1.1 in Section 1 above).

Proof. Let $P(v^0, v^1, \ldots, v^m, w^0, w^1, \ldots, w^n, z)$ be a polynomial expression, with complex coefficients, in $(1+m)+(1+n)+1$ indeterminates (or, equally well, in two differential indeterminates v, w and with coefficients in the field $\mathbb{C}(z)$). Assume that P is nullified by the pair χ, Γ according to

(A.72) $$P(\chi(z), \chi'(z), \ldots, \chi^{(m)}(z), \Gamma(z), \Gamma'(z), \ldots, \Gamma^{(n)}(z), z) \equiv 0, \text{ for } z \in \mathbb{C}.$$

Then we must demonstrate that the polynomial expression P is trivial, (all coefficients are zero, or, equally well, $P = 0$ as a polynomial function in $(1+m)+(1+n)+1$ independent complex variables).

We proceed by contradiction and suppose that there exists a non-trivial polynomial expression

(A.73) $$P(v^0, v^1, \ldots, v^m, w^0, w^1, \ldots, w^n, z)$$

satisfying the condition of (A.72), and we consider the set

(A.74) $$\mathfrak{M} = \{P(v, w, z), \text{ nontrivial polynomials satisfying (A.72)(A.73)}\}.$$

We shall suppose that $\mathfrak{M}$ is non-empty and from this derive a contradiction.

Following the discussions from (A.17) through (A.70), especially Definitions A.1, A.2, A.3, and Lemmas A.2, A.3, we consider the non-empty sets

(A.75) $$\mathfrak{M}_0 \subset \widehat{\mathfrak{M}} \subset \mathfrak{M},$$

and the corresponding minimal polynomial expression

(A.76) $$F_0(v, w, z) \in \widehat{\mathfrak{M}} \quad (\text{with height } (\widehat{j}, \widehat{k}))$$

which satisfies the Fundamental Functional Identity

(A.77) $$F_0(v, zw^0, zw^1 + w^0, \ldots, zw^{\widehat{n}} + \widehat{n}w^{\widehat{n}-1}, z+1) \equiv z^K F_0(v, w, z)$$

with

(A.78) $$K = \widehat{k}_0 + \widehat{k}_1 + \cdots + \widehat{k}_{\widehat{n}},$$

as in Lemma 3A, in particular (A.50), (A.51).

Now w^0 is not a factor of $F_0(v, w, z)$ and so some of the terms of $F_0(v, w, z)$ do not contain w^0 explicitly, moreover these terms are precisely the non-zero terms of $F_0(v, 0, w^1,\ w^2, \ldots, w^{\widehat{n}}, z)$. Hence $F_0(v, 0, w^1, w^2, \ldots, w^{\widehat{n}}, z)$ is a non-trivial polynomial and we now consider (A.77) with $w^0 = 0$, namely

(A.79) $F_0(v, 0, zw^1, zw^2 + 2w^1, \ldots, zw^{\widehat{n}} + \widehat{n}w^{\widehat{n}-1}, z+1) = z^K F_0(v, 0, w^1, \ldots, v^{\widehat{n}}, z)$.

Consider the leading term of the polynomial designated by either side of (A.79). For instance,

(A.80)
$$F_0(v, 0, w^1, w^2, \ldots, w^n, z) = C(z)(v^0)^{\mu_0} \ldots (v^m)^{\mu_m}(w^1)^{\nu_1}(w^2)^{\nu_2} \ldots (w^n)^{\nu_n} + \ldots$$

(note: $w^0 = 0$ does not change the order of the preceeding $w^n, w^{n-1}, \ldots, w^1$, and the subsequent $v^m, \ldots, v^0$, and here we write m for $\widehat{m}$, and n for $\widehat{n}$, for notational simplicity). Then

(A.81)
$$\begin{aligned} &F_0(v, 0, zw^1, zw^2 + 2w^1, \ldots, zw^n + nw^{n-1}, z+1) = \\ &\quad C(z+1)(v)^{\mu} z^{\nu_1 + \cdots + \nu_n}(w^1)^{\nu_1}(w^2)^{\nu_2} \ldots (w^n)^{\nu_n} + \ldots \ . \end{aligned}$$

Here the calculation for the leading term in (A.81) follows the same argument as (A.61) and (A.62) in Lemma A.3 above.

Then (A.79) shows that, referring to (A.80) and (A.81),

(A.82)
$$C(z+1)z^{\nu_1+\nu_2+\cdots+\nu_n} = z^K C(z).$$

Therefore

(A.83)
$$\nu_1 + \nu_2 + \cdots + \nu_n = K,$$

and the polynomial $C(z)$ satisfies

(A.84)
$$C(z+1) = C(z)\ , \quad \text{for } z \in \mathbb{C}.$$

From (A.84) it follows that $C(z)$ is a constant of $\mathbb{C}$,

(A.85)
$$C(z) = C \neq 0.$$

Then, from (A.80),

(A.86) $F_0(v, 0, w^1, \ldots, w^n, z) = (C)(v)^{\mu}(w^1)^{\nu_1} \ldots (w^n)^{\nu_n} +$ lower height terms.

Upon setting $z = 1$ in (A.86), we obtain,

(A.87)
$$F_0(v, 0, w^1, \ldots, w^n, 1) = (C)(v)^{\mu}(w^1)^{\nu_1} \ldots (w^n)^{\nu_n} + \ldots .$$

(with same lower height terms - but with z replaced by 1).

Hence (A.87) asserts that

(A.88)
$$F_0(v, 0, w^1, w^2, \ldots, w^n, 1)$$

is a non-trivial polynomial.

But the Fundamental Functional Identity (A.77), with $z = 0$, asserts

(A.89)
$$F_0(v, 0, w^0, 2w^1, \ldots, nw^{n-1}, 1) = 0\ .$$

Thus, (A.87) and (A.89) demonstrate that the same polynomial is non-trivial and also trivial. This establishes the contradiction denying the existence of the minimal polynomial F_0, and proving that the set $\mathfrak{M}$ of (A.74) necessarily is empty, as demanded in the theorem. ☐

The following two corollaries are now straightforward consequences of Theorem A.3.

Corollary A.1. *Let χ and Γ in Theorem A.3 be specified:*

$$\chi(z) = \zeta(\sin 2\pi z) \text{ , using the Riemann zeta function of (A.2),}$$

and

$$\Gamma(z), \qquad \text{Euler gamma function of (A.1).}$$

Then the pair

$$\zeta(\sin 2\pi z) \text{ and } \Gamma(z)$$

are diff-independent over $\mathbb{C}$ (or, equally well, over $\mathbb{C}(z)$ or $\mathbb{E}$).

Therefore

$$\text{(A.90)} \qquad \text{Diff-Trans } \partial^0 \quad \mathbb{C} < \zeta(\sin 2\pi z), \Gamma(z) > /\mathbb{C} = 2.$$

Corollary A.2. *Let χ and Γ in Theorem A.3 be specified:*

$$\chi(z) = \Gamma(\sin 2\pi z)$$

and

$$\Gamma(z), \qquad \text{Euler gamma function of (A.1).}$$

Then the pair

$$\Gamma(\sin 2\pi z) \text{ and } \Gamma(z)$$

are diff-independent over $\mathbb{C}$ (or, equally well, over $\mathbb{C}(z)$ or $\mathbb{E}$).

Therefore

$$\text{(A.91)} \qquad \text{Diff-Trans } \partial^0 \quad \mathbb{C} < \Gamma(\sin 2\pi z), \Gamma(z) > /\mathbb{C} = 2.$$

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